ALMOST COMPLETE CLUSTER TILTING OBJECTS IN GENERALIZED HIGHER CLUSTER CATEGORIES

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ABSTRACT. We study higher cluster tilting objects in generalized higher cluster categories arising from dg algebras of higher Calabi-Yau dimension. Taking advantage of silting mutations of Aihara-Iyama, we obtain a class of m-cluster tilting objects in generalized m-cluster categories. For generalized m-cluster categories arising from strongly $(m + 2)$-Calabi-Yau dg algebras, by using truncations of minimal cofibrant resolutions of simple modules, we prove that each almost complete m-cluster tilting P-object has exactly $m + 1$ complements with periodicity property. This leads us to the conjecture that each liftable almost complete m-cluster tilting object has exactly $m + 1$ complements in generalized m-cluster categories arising from m-rigid good completed deformed preprojective dg algebras.

1. INTRODUCTION

Cluster categories associated to acyclic quivers were introduced in [8], where the authors gave an additive categorification of the finite type cluster algebras introduced by Fomin and Zelevinsky [12] [13]. The cluster category of an acyclic quiver Q is defined as the orbit category of the derived category of finite dimensional representations of Q under the action of $\tau^{-1}\Sigma$, where τ is the AR-translation and Σ the suspension functor. If we replace the autoequivalence $\tau^{-1}\Sigma$ with $\tau^{-1}\Sigma^m$ for some integer $m \geq 2$, we obtain the m-cluster category, which was first mentioned and proved to be triangulated in [22], cf. also [29]. In the cluster category, the exchange relations of the corresponding cluster algebra are modeled by exchange triangles. It was shown in [18] that every almost complete cluster tilting object admits exactly two complements. In the higher cluster category, exchange triangles are replaced by AR-angles, whose existence (in the more general set up of Krull-Schmidt Hom-finite triangulated categories with Serre functors) was shown in [18]. Both [31] and [32] proved that each almost complete m-cluster tilting object has exactly $m + 1$ complements in an m-cluster category. In this paper, we study the analogous statements for almost complete m-cluster tilting objects in certain $(m+1)$-Calabi-Yau triangulated categories.

Amiot [2] constructed generalized cluster categories using 3-Calabi-Yau dg algebras which satisfy some suitable assumptions. A special class is formed by the generalized cluster categories associated to Ginzburg algebras [14] coming from suitable quivers with potentials. If the quiver is acyclic, the generalized cluster category is triangle equivalent to the classical cluster category. Amiot's results were extended by the author to generalized m-cluster categories in [16] by changing the Calabi-Yau dimension from 3 to $m + 2$ for an arbitrary positive integer m. As one of the applications, she particularly considered generalized higher cluster categories associated to Ginzburg dg categories [24] coming from suitable graded quivers with superpotentials.

In the representation theory of algebras, mutation plays an important role. Here we recall several kinds of mutation. Cluster algebras associated to finite quivers without loops or 2-cycles are defined using mutation of quivers. As an extension of quiver mutation, the mutation of quivers with potentials was introduced in [11]. Moreover, the mutation of decorated representations of quivers with potentials, which can be viewed as a generalization of the BGP construction, was also studied in [11]. Tilting modules over finite dimensional algebras are very nice objects, although some of them can not be mutated. In the cluster category associated to an acyclic quiver, mutation of cluster tilting objects is always possible [8]. It is determined by exchange

triangles and corresponds to mutation of clusters in the corresponding cluster algebra via a certain cluster character [10]. In the derived categories of finite dimensional hereditary algebras, a mutation operation was given in [9] on silting objects, which were first studied in [25]. Silting mutation of silting objects in triangulated categories, which is always possible, was investigated recently by Aihara and Iyama in [1].

The aim of this paper is to study higher cluster tilting objects in generalized higher cluster categories arising from dg algebras of higher Calabi-Yau dimension. Under certain assumptions on the dg algebras (Assumptions 2.1), tilting objects do not exist in the derived categories (Remark 2.6). Thus, we consider silting objects, e.g., the dg algebras themselves. The author was motivated by the construction of tilting complexes in Section 4 of [17].

This article is organized as follows: In Section 2, we list our assumptions on dg algebras and use the standard t-structure to situate the silting objects which are iteratively obtained from P-indecomposables with respect to the fundamental domain. In Section 3, using silting objects we construct higher cluster tilting objects in generalized higher cluster categories. We show that in such a category each liftable almost complete m-cluster tilting object has at least $m+1$ complements. In Section 4, we specialize to strongly higher Calabi-Yau dg algebras. By studying minimal cofibrant resolutions of simple modules of good completed deformed preprojective dg algebras, we obtain isomorphisms in generalized higher cluster categories between images of some left mutations and images of some right mutations of the same P-indecomposable. Using this, we derive the periodicity property of the images of iterated silting mutations of P-indecomposables in Section 5, where we also construct $(m+1)$-Calabi-Yau triangulated categories containing infinitely many indecomposable m-cluster tilting objects. We obtain an explicit description of the terms of Iyama-Yoshino's AR angles in this situation, and we deduce that each almost complete m-cluster tilting P-object in the generalized m-cluster category associated to a suitable completed deformed preprojective dg algebra has exactly $m+1$ complements in Section 6. We show that the truncated dg subalgebra at degree zero of the dg endomorphism algebra of a silting object in the derived category of a good completed deformed preprojective dg algebra is also strongly Calabi-Yau in Section 7. Then we conjecture a class (namely m-rigid) of good completed deformed preprojective dg algebras such that each liftable almost complete m-cluster tilting object should have exactly $m+1$ complements in the associated generalized m-cluster category. In Section 8, we give a long exact sequence to show the relations between extension spaces in generalized higher cluster categories and extension spaces in derived categories. This sequence generalizes the short exact sequence obtained by Amiot [2] in the 2-Calabi-Yau case. At the end, we show that any almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ is liftable if Π is the completed deformed preprojective dg algebra arising from an acyclic quiver.

Notation. For a collection $\mathcal{X}$ of objects in an additive category $\mathcal{T}$, we denote by $\mathrm{add}\mathcal{X}$ the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{X}$ and is closed under finite direct sums, summands and isomorphisms. Let k be an algebraically closed field of characteristic zero.

2. Silting objects in derived categories

Let A be a differential graded (for simplicity, write 'dg') k-algebra. We write $\mathrm{per}A$ for the *perfect derived category* of A, i.e. the smallest triangulated subcategory of the derived category $\mathcal{D}(A)$ containing A and stable under passage to direct summands. We denote by $\mathcal{D}_{fd}(A)$ the *finite dimensional derived category* of A whose objects are those of $\mathcal{D}(A)$ with finite dimensional total homology.

A dg k-algebra A is *pseudo-compact* if it is endowed with a complete separated topology which is generated by two-sided dg ideals of finite codimension. A (pseudo-compact) dg algebra

A is *(topologically) homologically smooth* if A lies in perA^e, where A^e is the (completed) tensor product of A^{op} and A over k. For example, suppose that A is of the form $(\widehat{kQ}, d)$, where $\widehat{kQ}$ is the completed path algebra of a finite graded quiver Q with respect to the two-sided ideal $\mathfrak{m}$ of $\widehat{kQ}$ generated by the arrows of Q, and the differential d takes each arrow of Q to an element of $\mathfrak{m}$; it was stated in [26] that A is pseudo-compact and topologically homologically smooth.

Assumptions 2.1. Let m be a positive integer. Suppose that A is a (pseudo-compact) dg k-algebra and has the following four additional properties:

  a) A is (topologically) homologically smooth;
  b) the pth homology $H^p A$ vanishes for each positive integer p;
  c) the zeroth homology $H^0 A$ is finite dimensional;
  d) A is $(m+2)$-Calabi-Yau as a bimodule, *i.e.*, there is an isomorphism in $\mathcal{D}(A^e)$

$$\mathrm{RHom}_{A^e}(A, A^e) \simeq \Sigma^{-m-2} A.$$

Theorem 2.2 ([24])**.** *(Completed) Ginzburg dg categories $\Gamma_{m+2}(Q, W)$ associated to graded quivers with superpotentials (Q, W) are (topologically) homologically smooth and $(m+2)$-Calabi-Yau.*

Lemma 2.3 ([23])**.** *Suppose that A is (topologically) homologically smooth. Then the category $\mathcal{D}_{fd}(A)$ is contained in per$A$. If moreover A is $(m+2)$-Calabi-Yau for some positive integer m, then for all objects L of $\mathcal{D}(A)$ and M of $\mathcal{D}_{fd}(A)$, we have a canonical isomorphism*

$$D\mathrm{Hom}_{\mathcal{D}(A)}(M, L) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(L, \Sigma^{m+2} M).$$

Throughout this paper, we always consider the dg algebras satisfying Assumptions 2.1.

Proposition 2.4 ([16])**.** *Under Assumptions 2.1, the triangulated category per$A$ is Hom-finite.*

Let $(\mathcal{D}A)^c$ denote the full subcategory of $\mathcal{D}(A)$ consisting of compact objects. Since each idempotent in $\mathcal{D}(A)$ is split and $(\mathcal{D}A)^c$ is closed under direct summands, each idempotent in $(\mathcal{D}A)^c$ is also split. Therefore, the category per$A$ which is equal to $(\mathcal{D}A)^c$ by [21] is a k-linear Hom-finite category with split idempotents. It follows that per$A$ is a Krull-Schmidt triangulated category.

Definitions 2.5. Let A be a dg algebra satisfying Assumptions 2.1.

  a) An object $X \in$ per$A$ is *silting* (resp. *tilting*) if per$A =$ thick$X$ the smallest thick subcategory of per$A$ containing X, and the spaces $\mathrm{Hom}_{\mathcal{D}(A)}(X, \Sigma^i X)$ are zero for all integers $i > 0$ (resp. $i \neq 0$).
  b) An object $Y \in$ per$A$ is *almost complete silting* if there is some indecomposable object Y' in $(\mathrm{per}A)\backslash(\mathrm{add}Y)$ such that $Y \oplus Y'$ is a silting object. Here Y' is called a *complement* of Y.

Clearly the dg algebra A itself is a silting object since the space $\mathrm{Hom}_{\mathcal{D}(A)}(A, \Sigma^i A)$ is isomorphic to $H^i A$ which is zero for each positive integer.

Remark 2.6. Under Assumptions 2.1, tilting objects do not exist in per$A$. Otherwise, let T be a tilting object in per$A$. By definition, the object T generates per$A$. Then for any object M in $\mathcal{D}(A)$, it belongs to the subcategory $\mathcal{D}_{fd}(A)$ if and only if $\sum_{p \in \mathbb{Z}} \dim\mathrm{Hom}_{\mathcal{D}(A)}(T, \Sigma^p M)$ is finite. Since the space $\mathrm{Hom}_{\mathcal{D}(A)}(T, T)$ is finite dimensional by Proposition 2.4 and the space $\mathrm{Hom}_{\mathcal{D}(A)}(T, \Sigma^p T)$ vanishes for any nonzero integer p, the object T belongs to $\mathcal{D}_{fd}(A)$. Note that $\mathcal{D}_{fd}(A)$ is $(m+2)$-Calabi-Yau as a triangulated category by Lemma 2.3. Thus, we have the following isomorphism

$$(0 =)\mathrm{Hom}_{\mathcal{D}(A)}(T, \Sigma^{m+2} T) \simeq D\mathrm{Hom}_{\mathcal{D}(A)}(T, T)(\neq 0).$$

Here we obtain a contradiction. Therefore, tilting objects do not exist.

Assume that H^0A is a basic algebra. Let e be a primitive idempotent element of H^0A. We denote by P the indecomposable direct summand eA (in the derived category $\mathcal{D}(A)$) of A and call it a *P-indecomposable*. We denote by M the dg module $(1-e)A$. It follows from Proposition 2.4 that the subcategory add$M$ is functorially finite [4] in add$A$. Let us write RA_0 for P (later we will also write LA_0 for P).

By induction on $t \geq 1$, we define RA_t as follows: take a minimal right (add$M$)-approximation $f^{(t)} : A^{(t)} \to RA_{t-1}$ of RA_{t-1} in $\mathcal{D}(A)$ and form the triangle in $\mathcal{D}(A)$

$$RA_t \xrightarrow{\alpha^{(t)}} A^{(t)} \xrightarrow{f^{(t)}} RA_{t-1} \longrightarrow \Sigma RA_t.$$

Dually, for each positive integer t, we take a minimal left (add$M$)-approximation $g^{(t)} : LA_{t-1} \to B^{(t)}$ of LA_{t-1} in $\mathcal{D}(A)$, and form the triangle in $\mathcal{D}(A)$

$$LA_{t-1} \xrightarrow{g^{(t)}} B^{(t)} \xrightarrow{\beta^{(t)}} LA_t \longrightarrow \Sigma LA_{t-1}.$$

The object RA_t is called the *right mutation* of RA_{t-1} (with respect to M), and LA_t is called the *left mutation* of LA_{t-1} (with respect to M).

Theorem 2.7 ([1]). *For each nonnegative integer t, the objects $M \oplus RA_t$ and $M \oplus LA_t$ are silting objects in* per$A$. *Moreover, any basic silting object containing M as a direct summand is either of the form $M \oplus RA_t$ or of the form $M \oplus LA_t$.*

From the construction and the above theorem, we know that the morphisms $\alpha^{(t)}$ (resp. $\beta^{(t)}$) are minimal left (resp. minimal right) (add$M$)-approximations in $\mathcal{D}(A)$ and that the objects RA_t and LA_t are indecomposable objects in $\mathcal{D}(A)$ which do not belong to add$M$.

We simply denote $\mathcal{D}(A)$ by $\mathcal{D}$. Let $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 1}$) be the full subcategory of $\mathcal{D}$ whose objects are the dg modules X such that H^pX vanishes for each positive (resp. nonpositive) integer p. For a complex X of k-modules, we denote by $\tau_{\leq 0}X$ the subcomplex with $(\tau_{\leq 0}X)^0 = ker d^0$, and $(\tau_{\leq 0}X)^i = X^i$ for negative integers i, otherwise zero. Set $\tau_{\geq 1}X = X/\tau_{\leq 0}X$.

Proposition 2.8. *For each integer $t \geq 0$, the object RA_t belongs to the subcategory $\mathcal{D}^{\leq t} \cap {}^\perp\mathcal{D}^{\leq -1} \cap$ per$A$, and the object LA_t belongs to the subcategory $\mathcal{D}^{\leq 0} \cap {}^\perp\mathcal{D}^{\leq -t-1} \cap$ per$A$.*

Proof. We consider the triangles appearing in the constructions of RA_t, and similarly for LA_t.

The object $RA_0(= P)$ belongs to $\mathcal{D}^{\leq 0} \cap {}^\perp\mathcal{D}^{\leq -1} \cap$ per$A$ since the dg algebra A has its homology concentrated in nonpositive degrees. The object RA_t is an extension of $A^{(t)}$ by $\Sigma^{-1}RA_{t-1}$, which both belong to the subcategory $\mathcal{D}^{\leq t} \cap$ per$A$. Thus, the object RA_t belongs to $\mathcal{D}^{\leq t} \cap$ per$A$. We do induction on t to show that RA_t belongs to ${}^\perp\mathcal{D}^{\leq -1}$. Let Y be an object in $\mathcal{D}^{\leq -1}$. By applying the functor $\mathrm{Hom}_\mathcal{D}(-, Y)$ to the triangle

$$RA_t \longrightarrow A^{(t)} \xrightarrow{f^{(t)}} RA_{t-1} \longrightarrow \Sigma RA_t,$$

we obtain the long exact sequence

$$\ldots \to \mathrm{Hom}_\mathcal{D}(A^{(t)}, Y) \to \mathrm{Hom}_\mathcal{D}(RA_t, Y) \to \mathrm{Hom}_\mathcal{D}(\Sigma^{-1}RA_{t-1}, Y) \to \ldots.$$

Since ΣY belongs to $\mathcal{D}^{\leq -2}$, by hypothesis, the space $\mathrm{Hom}_\mathcal{D}(\Sigma^{-1}RA_{t-1}, Y)$ is zero. Thus, the object RA_t belongs to ${}^\perp\mathcal{D}^{\leq -1}$. $\square$

Assume that $\{e_1, \cdots, e_n\}$ is a collection of primitive idempotent elements of H^0A. We denote by S_i the simple module corresponding to e_iA. For any object X in per$A$, we define the support of X as follows:

Definition 2.9. The *support* of X is defined as the set

$$supp\,(X) := \{j \in \mathbb{Z} | \mathrm{Hom}_\mathcal{D}(X, \Sigma^j S_i) \neq 0 \,\text{for some simple module}\, S_i\}.$$

Proposition 2.10. *For any nonnegative integer t, we have the following inclusions:*

1) $\{-t\} \subseteq supp\,(RA_t) \subseteq [-t, 0]$,

2) $\{t\} \subseteq supp\,(LA_t) \subseteq [0,t]$.

Proof. We only show the first statement, since the second one can be deduced in a similar way.

By Proposition 2.8, the object RA_t belongs to $\mathcal{D}^{\leq t} \cap {}^{\perp}\mathcal{D}^{\leq -1} \cap \mathrm{per}A$. Therefore, the space $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^r S_i)$ vanishes for each integer $r \geq 1$ since $\Sigma^r S_i$ lies in $\mathcal{D}^{\leq -1}$ and the space $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{r'} S_i)$ vanishes for each integer $r' \leq -t-1$ since $\Sigma^{r'} S_i$ lies in $\mathcal{D}^{\geq t+1}$. Thus, we have the inclusion $supp\,(RA_t) \subseteq [-t, 0]$.

Let S_P be the simple module corresponding to the P-indecomposable P from which RA_t and LA_t are obtained by mutation. We will show that $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{-t} S_P)$ is nonzero. Clearly, the space $\mathrm{Hom}_{\mathcal{D}}(P, S_P)$ is nonzero. We do induction on the integer t. Assume that the space $\mathrm{Hom}_{\mathcal{D}}(RA_{t-1}, \Sigma^{1-t} S_P)$ is nonzero. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, \Sigma^{1-t} S_P)$ to the triangle

$$RA_t \to A^{(t)} \to RA_{t-1} \to \Sigma RA_t,$$

where $A^{(t)}$ belongs to $(\mathrm{add}A)\backslash(\mathrm{add}P)$, we get the long exact sequence

$$\cdots \to (\Sigma A^{(t)}, \Sigma^{1-t} S_P) \to (\Sigma RA_t, \Sigma^{1-t} S_P) \to (RA_{t-1}, \Sigma^{1-t} S_P) \to (A^{(t)}, \Sigma^{1-t} S_P) \to \cdots,$$

where both the leftmost term and the rightmost term are zero. Therefore, $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{-t} S_P)$ is nonzero. This completes the proof. $\qquad\square$

Now we deduce the following corollary, which can also be deduced from Theorem 2.43 in [1].

Corollary 2.11.   1) *For any two nonnegative integers $r \neq t$, the object RA_r is not isomorphic to RA_t, and the object LA_r is not isomorphic to LA_t.*
     2) *For any two positive integers r and t, the objects RA_r and LA_t are not isomorphic.*

Proof. Assume that $r > t \geq 0$. Following Proposition 2.10, we have that

$$\mathrm{Hom}_{\mathcal{D}}(RA_r, \Sigma^{-r} S_P) \neq 0, \quad \text{while} \quad \mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{-r} S_P) = 0.$$

Thus, the objects RA_r and RA_t are not isomorphic. Similarly for LA_r and LA_t. Also in a similar way, we can obtain the second statement. $\qquad\square$

Combining Theorem 2.7 with Proposition 2.10, we can deduce the following corollary, which is analogous to Corollary 4.2 of [17]:

Corollary 2.12. *For any positive integer l, up to isomorphism, the object M admits exactly $2l-1$ complements whose supports are contained in $[1-l, l-1]$. These give rise to basic silting objects. They are the indecomposable objects RA_t and LA_t for $0 \leq t < l$.*

3. From silting objects to m-cluster tilting objects

Let $\mathcal{F}$ be the full subcategory $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -m-1} \cap \mathrm{per}A$ of $\mathcal{D}$. It is called the fundamental domain in [16]. Following Lemma 2.3, the category $\mathcal{D}_{fd}(A)$ is a triangulated thick subcategory of $\mathrm{per}A$. The triangulated quotient category $\mathcal{C}_A = \mathrm{per}A/\mathcal{D}_{fd}(A)$ is called the generalized m-cluster category [16]. We denote by π the canonical projection functor from $\mathrm{per}A$ to $\mathcal{C}_A$.

Proposition 3.1 ([16]). *Under Assumptions 2.1, the projection functor $\pi : \mathrm{per}A \longrightarrow \mathcal{C}_A$ induces a k-linear equivalence between $\mathcal{F}$ and $\mathcal{C}_A$.*

Theorem 3.2 ([16] Theorem 2.2, [26] Theorem 7.21). *If A satisfies Assumptions 2.1, then*

     1) *the generalized m-cluster category $\mathcal{C}_A$ is Hom-finite and $(m+1)$-Calabi-Yau;*
     2) *the object $T = \pi(A)$ is an m-cluster tilting object in $\mathcal{C}_A$, i.e.,*

$$\mathrm{add}T = \{L \in \mathcal{C}_A | \, \mathrm{Hom}_{\mathcal{C}_A}(T, \Sigma^r L) = 0, \, r = 1, \ldots, m\}.$$

Theorem 3.3. *The image of any silting object under the projection functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ is an m-cluster tilting object in $\mathcal{C}_A$.*

Proof. Assume that Z is an arbitrary silting object in per$A$. Without loss of generality, we can assume that Z is a cofibrant dg A-module [19]. We denote by Γ the dg endomorphism algebra $\mathrm{Hom}_A^\bullet(Z,Z)$. Since the spaces $\mathrm{Hom}_\mathcal{D}(Z,\Sigma^i Z)$ are zero for all positive integers i, the dg algebra Γ has its homology concentrated in nonpositive degrees. The zeroth homology of Γ is isomorphic to the space $\mathrm{Hom}_\mathcal{D}(Z,Z)$ which is finite dimensional by Proposition 2.4.

Since Z is a compact generator of $\mathcal{D}$, the left derived functor $F = - \overset{L}{\otimes}_\Gamma Z$ is a Morita equivalence [19] from $\mathcal{D}(\Gamma)$ to $\mathcal{D}$ which sends Γ to Z. Therefore, the dg algebra Γ is also (topologically) homologically smooth and $(m+2)$-Calabi-Yau. Thus, the generalized m-cluster category $\mathcal{C}_\Gamma$ is well defined. The equivalence F also induces a triangle equivalence from the generalized m-cluster category $\mathcal{C}_\Gamma$ to $\mathcal{C}_A$ which sends $\pi(\Gamma)$ to $\pi(Z)$. By Theorem 3.2, the image $\pi(\Gamma)$ is an m-cluster tilting object in $\mathcal{C}_\Gamma$. Hence, the image of Z is an m-cluster tilting object in $\mathcal{C}_A$. $\qquad\square$

In particular, for each nonnegative integer t, the images of $LA_t \oplus M$ and $RA_t \oplus M$ in the generalized m-cluster category $\mathcal{C}_A$ are m-cluster tilting objects.

Definitions 3.4. Let A be a dg algebra satisfying Assumptions 2.1 and $\mathcal{C}_A$ its generalized m-cluster category.

    a) An object X in $\mathcal{C}_A$ is called an *almost complete m-cluster tilting object* if there exists some indecomposable object X' in $\mathcal{C}_A \backslash$ (add$X$) such that $X \oplus X'$ is an m-cluster tilting object. Here X' is called a *complement* of X. In particular, we call $\pi(M)$ an *almost complete m-cluster tilting P-object*.

    b) An almost complete m-cluster tilting object Y is said to be *liftable* if there exists a basic silting object Z in per$A$ such the $\pi(Z/Z')$ is isomorphic to Y for some indecomposable direct summand Z' of Z.

Proposition 3.5. *Let A be a 3-Calabi-Yau dg algebra satisfying Assumptions 2.1. Then any $(1-)$cluster tilting object in $\mathcal{C}_A$ is induced by a silting object in $\mathcal{F}$ under the canonical projection π.*

Proof. Let T be a cluster tilting object in $\mathcal{C}_A$. By Proposition 3.1, we know that there exists an object Z in the fundamental domain $\mathcal{F}$ such that $\pi(Z) = T$.

First we will claim that Z is a partial silting object, that is, the spaces $\mathrm{Hom}_\mathcal{D}(Z,\Sigma^i Z)$ are zero for all positive integers i. Since Z belongs to $\mathcal{F}$, clearly these spaces vanish for all integers $i \geq 2$. Consider the case $i = 1$. The following short exact sequence

$$0 \to \mathrm{Ext}_\mathcal{D}^1(X,Y) \to \mathrm{Ext}_{\mathcal{C}_A}^1(X,Y) \to D\mathrm{Ext}_\mathcal{D}^1(Y,X) \to 0$$

was shown to exist in [2] for any objects X, Y in $\mathcal{F}$. We specialize both X and Y to the object Z. The middle term in the short exact sequence is zero since T is a cluster tilting object. Thus, the object Z is partial silting.

Second we will show that Z generates per$A$. Consider the following triangle

$$A \overset{f}{\to} Z_0 \to Y \to \Sigma A$$

in $\mathcal{D}$, where f is a minimal left (add$Z$)-approximation in $\mathcal{D}$. It is easy to see that Y also belongs to $\mathcal{F}$. Therefore, the above triangle can be viewed as a triangle in $\mathcal{C}_A$ with f a minimal left (add$Z$)-approximation in $\mathcal{C}_A$. Applying the functor $\mathrm{Hom}_{\mathcal{C}_A}(-,Z)$ to the triangle, we get the exact sequence

$$\mathrm{Hom}_{\mathcal{C}_A}(Z_0,Z) \to \mathrm{Hom}_{\mathcal{C}_A}(A,Z) \to \mathrm{Hom}_{\mathcal{C}_A}(\Sigma^{-1}Y,Z) \to \mathrm{Hom}_{\mathcal{C}_A}(\Sigma^{-1}Z_0,Z)$$

Therefore the space $\mathrm{Hom}_{\mathcal{C}_A}(Y,\Sigma Z)$ becomes zero. As a consequence, Y belongs to add$Z$ in $\mathcal{C}_A$. Since both Y and Z are in $\mathcal{F}$, the object Y also belongs to add$Z$ in $\mathcal{D}$. Therefore, the dg algebra A belongs to the subcategory thick$Z$ of per$A$. It follows that Z generates per$A$. $\qquad\square$

Theorem 3.6. *The almost complete m-cluster tilting P-object $\pi(M)$ has at least $m + 1$ complements in $\mathcal{C}_A$.*

Proof. Following Proposition 2.8 and Corollary 2.11, the pairwise non isomorphic indecomposable objects LA_t $(0 \leq t \leq m)$ belong to the fundamental domain $\mathcal{F}$. Therefore, by Proposition 3.1, the $m+1$ objects $\pi(P), \pi(LA_1), \ldots, \pi(LA_m)$ are indecomposable and pairwise non isomorphic in $\mathcal{C}_A$. It follows that $\pi(M)$ has at least $m+1$ complements in $\mathcal{C}_A$. $\qquad\square$

Let us generalize the above theorem:

Theorem 3.7. *Each liftable almost complete m-cluster tilting object has at least $m + 1$ complements in $\mathcal{C}_A$.*

Proof. Let Y be a liftable almost complete m-cluster tilting object. By definition there exists a basic silting object Z (assume that Z is cofibrant) in per$A$ such the $\pi(Z/Z')$ is isomorphic to Y for some indecomposable direct summand Z' of Z. Let Γ be the dg endomorphism algebra $\mathrm{Hom}_A^\bullet(Z, Z)$. Then $H^0\Gamma$ is a basic algebra.

Similarly as in the proof of Theorem 3.3, the dg algebra Γ satisfies Assumptions 2.1, and the left derived functor $F := - \overset{L}{\otimes}_\Gamma Z$ induces a triangle equivalence from $\mathcal{C}_\Gamma$ to $\mathcal{C}_A$ which sends $\pi(\Gamma)$ to $\pi(Z)$. Let Γ' be the object $\mathrm{Hom}_A^\bullet(Z, Z/Z')$ in perΓ. Then $\pi(\Gamma')$ is the almost complete m-cluster tilting P-object in $\mathcal{C}_\Gamma$ which corresponds to Y under the functor F. It follows from Theorem 3.6 that $\pi(\Gamma')$ has at least $m + 1$-complements in $\mathcal{C}_\Gamma$. So does the liftable almost complete m-cluster tilting object Y in $\mathcal{C}_A$. $\qquad\square$

Remark 3.8. Let $\mathcal{T}$ be a Krull-Schmidt Hom-finite triangulated category with a Serre functor. In fact, following [18], one can get that any almost complete m-cluster tilting object Y in $\mathcal{T}$ has at least $m + 1$ complements. Note that the notation in [16] and [18] has some differences with each other, for example, m-cluster tilting objects in [16] correspond to $(m + 1)$-cluster tilting subcategories (or objects) in [18]. Here we use the same notation as [16]. Set $\mathcal{Y} = \mathrm{add}Y$, $\mathcal{Z} = \cap_{i=1}^m {}^\perp(\Sigma^i \mathcal{Y})$ and $\mathcal{U} = \mathcal{Z}/\mathcal{Y}$. Let X be an m-cluster tilting object in $\mathcal{T}$ which contains Y as a direct summand. Set $\mathcal{X} = \mathrm{add}X$. Then by Theorem 4.9 in [18], the subcategory $\mathcal{L} := \mathcal{X}/\mathcal{Y}$ is m-cluster tilting in the triangulated category $\mathcal{U}$. The subcategories $\mathcal{L}, \mathcal{L}\langle 1 \rangle, \ldots, \mathcal{L}\langle m \rangle$ are distinct m-cluster tilting subcategories of $\mathcal{U}$, where $\langle 1 \rangle$ is the shift functor in the triangulated category $\mathcal{U}$. Also by the same theorem, the one-one correspondence implies that the number of m-cluster tilting objects of $\mathcal{T}$ containing Y as a direct summand is at least $m + 1$.

4. Minimal cofibrant resolutions of simple modules for strongly $(m + 2)$-Calabi-Yau case

The well-known Connes long exact sequence (SBI-sequence) for cyclic homology [27] associated to a dg algebra A is as follows

$$\ldots \to HH_{m+3}(A) \overset{I}{\to} HC_{m+3}(A) \overset{S}{\to} HC_{m+1}(A) \overset{B}{\to} HH_{m+2}(A) \overset{I}{\to} \ldots,$$

where $HH_*(A)$ denotes the Hochschild homology of A and $HC_*(A)$ denotes the cyclic homology.

Let M and N be two dg A-modules with M in perA^e. Then in $\mathcal{D}(k)$ we have the isomorphism

$$\mathrm{RHom}_{A^e}(\mathrm{RHom}_{A^e}(M, A^e), N) \simeq M \overset{L}{\otimes}_{A^e} N.$$

An element $\xi = \sum_{i=1}^s \xi_{1i} \otimes \xi_{2i} \in H^r(M \overset{L}{\otimes}_{A^e} N)$ is *non-degenerate* if the corresponding map

$$\xi^+ : \mathrm{RHom}_{A^e}(M, A^e) \to \Sigma^r N$$

given by $\xi^+(\phi) = \sum_{i=1}^s (-1)^{|\phi||\xi|} \phi(\xi_{1i})_2 \xi_{2i} \phi(\xi_{1i})_1$ is an isomorphism. Throughout this article, we write $|\cdot|$ to denote the degrees.

Let l be a finite dimensional separable k-algebra. We fix a trace $Tr : l \to k$ and let $\sigma' \otimes \sigma''$ be the corresponding Casimir element (i.e., $\sigma' \otimes \sigma'' = \sum \sigma_i' \otimes \sigma_i''$ and $Tr(\sigma_i' \sigma_j'') = \delta_{ij}$). An *augmented dg l-algebra* is a dg algebra A equipped with dg k-algebra homomorphisms $l \overset{\varsigma}{\to} A \overset{\epsilon}{\to} l$ such that $\epsilon\varsigma$ is the identity. Following [30] we write $PCAlgc(l)$ for the category of pseudo-compact

augmented dg l-algebras satisfying $ker(\epsilon) = coker(\varsigma) = \mathrm{rad}A$. When forgetting the grading, $\mathrm{rad}A$ is just the Jacobson radical of the underlying ungraded algebra $A^u := \prod_r A^r$ of the dg algebra $A = (A^r)_r$.

The SBI-sequence can be extended to the case that $A \in PCAlgc(l)$, where $HH_*(A)(= H_*(A \overset{L}{\otimes}_{A^e} A))$ is computed by the pseudo-compact Hochschild complex. For more details, see section 8 and Appendix B in [30].

Definition 4.1 ([30]). An algebra $A \in PCAlgc(l)$ is *strongly $(m+2)$-Calabi-Yau* if A is topologically homologically smooth and $HC_{m+1}(A)$ contains an element η such that $B\eta$ is non-degenerate in $HH_{m+2}(A)$.

Theorem 4.2 ([30]). *Let $A \in PCAlgc(l)$. Assume that $A = (A^r)_{r \leq 0}$ is concentrated in nonpositive degrees. Then A is strongly $(m+2)$-Calabi-Yau if and only if there is a quasi-isomorphism $(\widehat{T_l V}, d) \to A$ as augmented dg l-algebras with V having the following properties*

  a) $d(V) \cap V = 0$;
  b) $V = V_c \oplus lz$ *with z an l-central element of degree $-m-1$, V_c finite dimensional and concentrated in degrees $[-m, 0]$;*
  c) $dz = \sigma'\eta\sigma''$ *with $\eta \in V_c \otimes_{l^e} V_c$ non-degenerate and antisymmetric under the flip F : $v_1 \otimes v_2 \to (-1)^{|v_1||v_2|} v_2 \otimes v_1$ for any v_1, v_2 in V_c.*

We would like to present the explicit construction of Ginzburg dg categories in the following straightforward proposition.

Proposition 4.3. *The completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q, W)$ associated to a finite graded quiver Q concentrated in degrees $[-m, 0]$ and a reduced superpotential W being a linear combination of paths of Q of degree $1 - m$ and of length at least 3, is strongly $(m+2)$-Calabi-Yau.*

Proof. We only need to check that $\widehat{\Gamma}_{m+2}(Q, W)$ satisfies the assumptions and condition 2) in Theorem 4.2 from its definition.

Let l be the separable k-algebra $\prod_{i \in Q_0} ke_i$. Let $\overline{Q}^G$ be the double quiver obtained from Q by adjoining opposite arrows a^* of degree $-m - |a|$ for arrows $a \in Q_1$. Let $\widetilde{Q}^G$ be obtained from $\overline{Q}^G$ by adjoining a loop t_i of degree $-m - 1$ for each vertex i. Then the completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q, W)$ is the completed path category $\widehat{T_l(\widetilde{Q}^G)}$ with the following differential

$$d(a) = 0, \quad a \in Q_1;$$
$$d(t_i) = e_i(\sum_{a \in Q_1}[a, a^*])e_i, \quad i \in Q_0;$$
$$d(a^*) = (-1)^{|a|}\frac{\partial W}{\partial a} = (-1)^{|a|}\sum_{p=uav}(-1)^{(|a|+|v|)|u|}vu, \quad a \in Q_1;$$

where the sum in the third formula runs over all homogeneous summands $p = uav$ of W.

Thus, the components of $\widehat{\Gamma}_{m+2}(Q, W)$ are concentrated in nonpositive degrees and $\widehat{\Gamma}_{m+2}(Q, W)$ $(= l \oplus \prod_{s \geq 1}(\widetilde{Q}^G)^{\otimes_l s})$ lies in $PCAlgc(l)$.

The differential above which is induced by the reduced superpotential W satisfies that $d(\widetilde{Q}^G) \cap \widetilde{Q}^G = 0$. Set $z = \sum_{i \in Q_0} t_i$. Then z is an l-central element of degree $-m - 1$. Clearly, $\widetilde{Q}^G = \overline{Q}^G \oplus lz$, the double quiver $\overline{Q}^G$ is finite and concentrated in degrees $[-m, 0]$, and the element $d(z) = \sum_{a \in Q_1}(aa^* - (-1)^{|a||a^*|}a^*a)$ is antisymmetric under the flip F.

The last step is to show that $\eta := \sum_{a \in Q_1}[a, a^*]$ is non-degenerate, that is, the corresponding map

$$\eta^+ : \mathrm{Hom}_{l^e}(\overline{Q}^G, l^e) \longrightarrow \overline{Q}^G, \quad \phi \to (-1)^{|\phi||\eta|}\phi(\eta_1)_2 \eta_2 \phi(\eta_1)_1$$

is an isomorphism. Define morphisms $\phi_\gamma(\gamma \in \overline{Q}^G) : \overline{Q}^G \to l^e$ as follows

$$\phi_\gamma(\alpha) = \delta_{\alpha\gamma}e_{t(\alpha)} \otimes e_{s(\alpha)}.$$

Then $\{\phi_\gamma | \gamma \in \overline{Q}^G\}$ is a basis of the space $\mathrm{Hom}_{l^e}(\overline{Q}^G, l^e)$. Applying the map η^+, we obtain the images $\eta^+(\phi_a) = (-1)^{m|a|}a^*$ and $\eta^+(\phi_{a^*}) = (-1)^{1+|a^*|^2}a$ for arrows $a \in Q_1$. Thus, $\{\eta^+(\phi_\gamma)|\gamma \in \overline{Q}^G\}$ is a basis of $\overline{Q}^G$. Therefore, the element η is non-degenerate. $\qquad\square$

Now we write down the explicit construction of deformed preprojective dg algebras as described in [30]. Let Q be a finite graded quiver and L the subset of Q_1 consisting of all loops a of odd degree such that $|a| = -m/2$. Let $\overline{Q}^V$ be the double quiver obtained from Q by adjoining opposite arrows a^* of degree $-m - |a|$ for $a \in Q_1 \setminus L$ and putting $a^* = a$ without adjoining an extra arrow for $a \in L$. Let N be the Lie algebra $k\overline{Q}^V/[k\overline{Q}^V, k\overline{Q}^V]$ endowed with the necklace bracket $\{-, -\}$ (cf. [5], [15]). Let W be a superpotential which is a linear combination of homogeneous elements of degree $1 - m$ in N and satisfies $\{W, W\} = 0$ (in order to make the differential well-defined). Let $\widetilde{Q}^V$ be obtained from $\overline{Q}^V$ by adjoining a loop t_i of degree $-m - 1$ for each vertex i. Then the *deformed preprojective dg algebra* $\Pi(Q, m + 2, W)$ is the dg algebra $(k\widetilde{Q}^V, d)$ with the differential

$$da = \{W, a\} = (-1)^{(|a|+1)|a^*|}\frac{\partial W}{\partial a^*} = (-1)^{(|a|+1)|a^*|}\sum_{p=ua^*v}(-1)^{(|a^*|+|v|)|u|}vu;$$
$$da^* = \{W, a^*\} = (-1)^{|a|+1}\frac{\partial W}{\partial a} = (-1)^{|a|+1}\sum_{p=uav}(-1)^{(|a|+|v|)|u|}vu;$$
$$dt_i = e_i(\textstyle\sum_{a \in Q_1}[a, a^*])e_i;$$

where $a \in Q_1$ and $i \in Q_0$. Later we will denote the homogeneous elements rvu ($r \in k$) appearing in $d\alpha$ ($\alpha \in \overline{Q}^V$) by $y(\alpha, v, u)$.

Remark 4.4. As in Proposition 4.3, we see that the completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m + 2, W)$ associated to a finite graded quiver Q concentrated in degrees $[-m, 0]$ and a reduced superpotential W being a linear combination of paths of $\overline{Q}^V$ of length at least 3, is also strongly $(m + 2)$-Calabi-Yau.

Suppose that -1 is a square in the field k and denote by $\sqrt{-1}$ a chosen square root. Then the class of deformed preprojective dg algebras is strictly greater than the class of Ginzburg dg categories. Suppose that Q does not contain *special loops* (i.e., loops of odd degree which is equal to $-m/2$). Then we can easily see that $\Gamma_{m+2}(Q, W) = \Pi(Q, m + 2, -W)$. Otherwise, let Q^0 be the subquiver of Q obtained by removing the special loops. For each special loop a in Q_1, we draw a pair of loops a' and a'' which are also special at the same vertex of Q^0. Denote the new quiver by Q'. Let W' be the superpotential obtained from W by replacing each special loop by the corresponding element $a' + a''\sqrt{-1}$. Now we define a map $\iota : \Gamma_{m+2}(Q, W) \to \Pi(Q', m + 2, -W')$, which sends each special loop a of Q_1 to the element $a' + a''\sqrt{-1}$ and its dual a^* to the element $a' - a''\sqrt{-1}$ in $\Pi(Q', m + 2, -W')$, and is the identity on the other arrows of $\widetilde{Q}^G$. Then it is not hard to check that ι is a dg algebra isomorphism. It follows that Ginzburg dg categories are deformed preprojective dg algebras. For the strictness, see the following example.

Example 4.5. Suppose that m is 2. Let Q be the quiver consisting of only one vertex '•' and one loop a of degree -1. Then the Ginzburg dg category $\Gamma_4(Q, 0)$ and the deformed preprojective dg algebra $\Pi(Q, 4, 0)$ respectively have the the following underlying graded quivers

where $|a| = |a^*| = -1$ and $|t| = -3$. The differential takes the following values

$$d(a) = 0 = d(a^*), \quad d_{\Gamma_4(Q,0)}(t) = aa^* + a^*a, \quad d_{\Pi(Q,4,0)}(t) = 2a^2.$$

Then $\dim H^{-1}(\Gamma_4(Q, 0)) = 2$ while $\dim H^{-1}(\Pi(Q, 4, 0)) = 1$. Hence, these two dg algebras are not quasi-isomorphic. Moreover, it is obvious that the dg algebra $\Pi(Q, 4, 0)$ can not be realized as a Ginzburg dg category.

Lemma 4.6. *Let* $\Pi = \widehat{\Pi}(Q, m+2, W)$ *be a completed deformed preprojective dg algebra. Let x (resp. y) denote the minimal (resp. maximal) degree of the arrows of $\overline{Q}^V$. Then there exist a canonical completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', m+2, W')$ isomorphic to Π as a dg algebra, where the quiver Q' is concentrated in degrees $[-m/2, y]$.*

Proof. We can construct directly a quiver Q' and a superpotential W'.

We claim first that $x + y = -m$. Let x_1 (resp. y_1) denote the minimal (resp. maximal) degree of the arrows of Q. Then $\overline{Q}^V \setminus Q$ is concentrated in degrees $[-m - y_1, -m - x_1]$. If $x_1 \leq -m - y_1$, then $x = x_1$ and $y_1 \leq -m - x_1$. Hence, $x + y = x_1 + (-m - x_1) = -m$. Similarly for the case '$-m - y_1 \leq x_1$'.

Let Q^0 be the subquiver of Q which has the same vertices as Q and whose arrows are those of Q with degree belonging to $[-m/2, y]$ $(= [(x + y)/2, y])$. In this case $|a^*| = -m - |a| \in [-m - y, -m/2] = [x, -m/2]$. For each arrow b of Q whose dual b^* has degree in $(-m/2, y]$, we add a corresponding arrow b' to Q^0 with the same degree as b^*. Denote the new quiver by Q'. Therefore, the quiver Q' has arrow set

$$\{a \in Q_1 \mid |a| \in [-m/2, y]\} \cup \{b' \mid |b'| = |b^*|, b \in Q_1 \text{ and } |b^*| \in (-m/2, y]\}.$$

We define a map $\iota : \widetilde{Q}^V \to \widetilde{Q'}^V$ by setting

$$\iota(a) = a,\ \iota(a^*) = a^*;\quad \iota(t_i) = t_i;\quad \iota(b) = (-1)^{|b||b^*|+1}b'^*,\ \iota(b^*) = b'.$$

Let W' be the superpotential obtained from W by replacing each arrow α in W by $\iota(\alpha)$. Then it is not hard to check that the map ι can be extended to a dg algebra isomorphism from Π to Π'. $\qquad\square$

In particular, if Q is concentrated in degrees $[-m, 0]$, then by the above lemma, the new quiver Q' is concentrated in degrees $[-m/2, 0]$. If the following two conditions

V1) Q a finite graded quiver concentrated in degrees $[-m/2, 0]$,

V2) W a reduced superpotential being a linear combination of paths of $\overline{Q}^V$ of degree $1 - m$ and of length ≥ 3,

hold, then we will say that the completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ is *good*.

Theorem 4.7 ([30]). *Let A be a strongly $(m+2)$-Calabi-Yau dg algebra with components concentrated in degrees ≤ 0. Suppose that A lies in $PCAlgc(l)$ for some finite dimensional separable commutative k-algebra l. Then A is quasi-isomorphic to some good completed deformed preprojective dg algebra.*

We consider the strongly $(m+2)$-Calabi-Yau case in this section, by Theorem 4.7, it suffices to consider good completed deformed preprojective dg algebras $\Pi = \widehat{\Pi}(Q, m+2, W)$. The simple Π-module S_i (attached to a vertex i of Q) belongs to the finite dimensional derived category $\mathcal{D}_{fd}(\Pi)$, hence it also belongs to perΠ. We will give a precise description of the objects RA_t and LA_t obtained from iterated mutations of a P-indecomposable $e_i\Pi$, where e_i is the primitive idempotent element associated to a vertex i of Q.

Definition 4.8 ([28]). Let $A = (\widehat{kQ}, d)$ be a dg algebra, where Q is a finite graded quiver and d is a differential sending each arrow to a (possibly infinite) linear combination of paths of length ≥ 1. A dg A-module M is *minimal perfect* if

a) its underlying graded module is of the form $\oplus_{j=1}^{N} R_j$, where R_j is a finite direct sum of shifted copies of direct summands of A, and

b) its differential is of the form $d_{int} + \delta$, where d_{int} is the direct sum of the differentials of these R_j $(1 \leq j \leq N)$, and δ, as a degree 1 map from $\oplus_{j=1}^{N} R_j$ to itself, is a strictly upper triangular matrix whose entries are in the ideal $\mathfrak{m}$ of A generated by the arrows of Q.

Lemma 4.9 ([28]). *Let M be a dg $A(=(\widehat{kQ},d))$-module such that M lies in $\mathrm{per}A$. Then M is quasi-isomorphic to a minimal perfect dg A-module.*

In the second part of this section, we illustrate how to obtain minimal perfect dg modules which are quasi-isomorphic to simple Π-modules from cofibrant resolutions [26]. If a cofibrant resolution $\mathbf{p}X$ of a dg module X is minimal perfect, then we say $\mathbf{p}X$ a *minimal cofibrant resolution* of X.

Let i be a vertex of Q and $P_i = e_i\Pi$. Consider the short exact sequence in the category $\mathcal{C}(\Pi)$ of dg modules
$$0 \to ker(p) \xrightarrow{\iota} P_i \xrightarrow{p} S_i \to 0,$$
where in the category $\mathrm{Grmod}(\Pi)$ of graded modules $ker(p)$ is the direct sum of $\rho P_{s(\rho)}$ over all arrows $\rho \in \widetilde{Q}_1^V$ with $t(\rho) = i$. The simple module S_i is quasi-isomorphic to $cone(ker(p) \xrightarrow{\iota} P_i)$, *i.e.*, the dg module
$$X = (\underline{X} = P_i \oplus \Sigma X_0' \oplus \ldots \oplus \Sigma X_{m+1}', d_X = \begin{pmatrix} d_{P_i} & \iota \\ 0 & -d_{ker(p)} \end{pmatrix}),$$
where for each integer $0 \le j \le m+1$, the object X_j' is the direct sum of $\rho P_{s(\rho)}$ ranging over all arrows $\rho \in \widetilde{Q}_1^V$ with $t(\rho) = i$ and $|\rho| = -j$. By Section 2.14 in [26], the dg module X is a cofibrant resolution of the simple module S_i.

Now let $P_j'\,(0 \le j \le m+1)$ be the direct sum of $P_{s(\rho)}$ where ρ ranges over all arrows in $\widetilde{Q}_1^V$ satisfying $t(\rho) = i$ and $|\rho| = -j$. Clearly, $P_{m+1}' = P_i$. We require that the ordering of direct summands $P_{s(\rho)}$ in P_j' is the same as the ordering of direct summands $\rho P_{s(\rho)}$ in X_j' for each integer $0 \le j \le m+1$. Let Y be an object whose underlying graded module is $\underline{Y} = P_i \oplus \Sigma P_0' \oplus \Sigma^2 P_1' \oplus \ldots \oplus \Sigma^{m+2} P_{m+1}'$. We endow $\underline{Y}$ with the degree 1 graded endomorphism $d_{int} + \delta_Y$, where d_{int} is the same notation as in Definition 4.8. The columns of δ_Y have the following two types: $(\alpha, 0, \ldots, -y_{red}(\alpha, v, u), \ldots, 0)^t$, and $(t_i, \ldots, -a^*, \ldots, (-1)^{|b||b^*|}b, \ldots, 0)^t$ for the last column. Here α is an arrow in $\overline{Q}^V$, while a is an arrow in Q and b is an arrow in $\overline{Q}^V \setminus Q$. Here $y_{red}(\alpha, v, u)$ is obtained from the path $y(\alpha, v, u) = \beta_s \ldots \beta_1$ (this notation is defined just before Remark 4.4) by removing the factor β_s. The ordering of the elements in each column is determined by the ordering of Y.

Let $f : Y \to X$ be a map constructed as the diagonal matrix whose elements are all arrows in $\widetilde{Q}_1^V$ with target at i, together with e_i as the first element. Moreover, we require that the ordering of these arrows is determined by Y (hence also by X), that is, the components of f are of the form
$$f_\rho : \Sigma^{|\rho|+1} P_{s(\rho)} \longrightarrow \Sigma \rho P_{s(\rho)}, \qquad u \mapsto \rho u.$$
It is not hard to check the identity $f(d_{int} + \delta_Y) = d_X f$. Hence, the morphism f is an isomorphism in $\mathcal{C}(\Pi)$, and the map $d_{int} + \delta_Y$ makes the object Y into a dg module which is minimal perfect. Therefore, the dg module Y is a minimal cofibrant resolution of the simple module S_i.

In the third part of this section, we show that when there are no loops of Q at vertex i, the truncations of the minimal cofibrant resolution Y of the simple module S_i produce RA_t and $LA_t\,(0 \le t \le m+1)$ obtained from the P-indecomposable P_i by iterated mutations. If we write M for the dg module Π/P_i, then the dg modules $P_j'\,(0 \le j \le m)$ appearing in Y lie in $\mathrm{add}M$. Let $\varepsilon_{\le t}Y$ be the submodule of Y with the inherited differential whose underlying graded module is the direct sum of those summands of Y with copies of shift $\le t$. Let $\varepsilon_{\ge t+1}Y$ be the quotient module $Y/(\varepsilon_{\le t}Y)$. Notice that $\varepsilon_{\le t}Y$ is a truncation of Y for the canonical weight structure on $\mathrm{per}\Pi$, cf. Bondarko, Keller-Nicolas.

Proposition 4.10. *Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ and i a vertex of Q. Assume that there are no loops of Q at vertex i. Then the following two isomorphisms*
$$\Sigma^{-t}\varepsilon_{\le t}Y \simeq RA_t \quad and \quad \Sigma^{-t-1}\varepsilon_{\ge t+1}Y \simeq LA_{m+1-t}$$
hold in the derived category $\mathcal{D} := \mathcal{D}(\Pi)$ for each integer $0 \le t \le m+1$.

Proof. We only consider the first isomorphism. Then the second one can be obtained dually. For arrows of $\overline{Q}^V$ of degree $-j$ ending at vertex i, we write α_j; for the symbols $-y_{red}(\alpha, v, u)$ of degree $-j$, we simply write $-y_{red}^j$, and for morphisms f of degree $-j$, we write f_j, where $0 \leq j \leq m$. Moreover, we use the notation $[x]$ to denote a matrix whose entries x have the same 'type' (in some obvious sense).

Clearly, when $t = 0$, we have that $\varepsilon_{\leq 0} Y = P_i = RA_0$.

When $t = 1$, we have the following isomorphisms

$$\Sigma^{-1}\varepsilon_{\leq 1}Y \simeq \left(\Sigma^{-1}P_i \oplus P_0', \begin{pmatrix} d_{\Sigma^{-1}P_i} & -[\alpha_0] \\ 0 & d_{P_0'} \end{pmatrix}\right) \simeq \Sigma^{-1}cone(P_0' \xrightarrow{h^{(1)}} P_i),$$

where each component of $h^{(1)}(= [\alpha_0])$ is the left multiplication by some α_0. Since W is reduced, the left multiplication by α_0 is nonzero in the space $\mathrm{Hom}_{\mathcal{D}}(P_0', P_i)$. Moreover, only the trivial paths e_i have zero degree, and there are no loops of $\overline{Q}^V$ of degree zero at vertex i. It follows that $h^{(1)}$ is a minimal right $(\mathrm{add}M)$-approximation of P_i. Then $\Sigma^{-1}\varepsilon_{\leq 1}Y$ and RA_1 are isomorphic in $\mathcal{D}$.

In general, assume that $\Sigma^{-t}\varepsilon_{\leq t}Y \simeq RA_t \, (1 \leq t \leq m)$. We will show that $\Sigma^{-t-1}\varepsilon_{\leq t+1}Y \simeq RA_{t+1}$. First we have the following isomorphism

$$\Sigma^{-t-1}\varepsilon_{\leq t+1}Y \simeq (\Sigma^{-t-1}P_i \oplus \Sigma^{-t}P_0' \oplus \ldots \oplus P_t',$$

$$\begin{pmatrix} d_{\Sigma^{-t-1}P_i} & (-1)^{t+1}[\alpha_0] & \cdots & (-1)^{t+1}[\alpha_{t-1}] & (-1)^{t+1}[\alpha_t] \\ 0 & d_{\Sigma^{-t}P_0'} & \cdots & (-1)^t[y_{red}^{t-2}] & (-1)^t[y_{red}^{t-1}] \\ & & \cdots & \cdots & \\ 0 & 0 & \cdots & d_{\Sigma^{-1}P_{t-1}'} & (-1)^t[y_{red}^0] \\ 0 & 0 & \cdots & 0 & d_{P_t'} \end{pmatrix})$$

$$\simeq \Sigma^{-1}cone(P_t' \xrightarrow{h^{(t+1)}} RA_t),$$

where $h^{(t+1)} = ((-1)^t[\alpha_t], (-1)^{t-1}[y_{red}^{t-1}], \ldots, (-1)^{t-1}[y_{red}^0])^t$. Each component of $h^{(t+1)}$ is a nonzero morphism in $\mathrm{Hom}_{\mathcal{D}}(P_t', RA_t)$, since the superpotential W is reduced. Otherwise, the arrow α_t will be a linear combination of paths of length ≥ 2. It follows that $h^{(t+1)}$ is right minimal. Let L be an arbitrary indecomposable object in $\mathrm{add}A$ and $f = (f_t, [f_{t-1}], \ldots, [f_1], [f_0])^t$ an arbitrary morphism in $\mathrm{Hom}_{\mathcal{D}}(L, RA_t)$. Then the vanishing of $d(f)$ implies that $d(f_t) = -[\alpha_0][f_{t-1}] - \ldots - [\alpha_{t-2}][f_1] - [\alpha_{t-1}][f_0]$. Since there are no loops of $\overline{Q}^V$ of degree $-t$ at vertex i, the map f_t which is homogeneous of degree $-t$ is a linear combination of the following forms:

(*i*) $f_t = \alpha_t g_0$, where $|g_0| = 0$. In this case, the differential

$$d(f_t) = d(\alpha_t g_0) = d(\alpha_t)g_0 = [\alpha_0][y_{red}^{t-1}]g_0 + \ldots + [\alpha_{t-1}][y_{red}^0]g_0,$$

which implies that $[f_r]$ is equal to $-[y_{red}^r]g_0 \, (0 \leq r \leq t-1)$. Then the equalities

$$f = \begin{pmatrix} f_t \\ [f_{t-1}] \\ \ldots \\ [f_1] \\ [f_0] \end{pmatrix} = \begin{pmatrix} \alpha_t g_0 \\ -[y_{red}^{t-1}]g_0 \\ \ldots \\ -[y_{red}^1]g_0 \\ -[y_{red}^0]g_0 \end{pmatrix} = \begin{pmatrix} (-1)^t\alpha_t \\ (-1)^{t-1}[y_{red}^{t-1}] \\ \ldots \\ (-1)^{t-1}[y_{red}^1] \\ (-1)^{t-1}[y_{red}^0] \end{pmatrix} (-1)^t g_0.$$

hold. Thus, the morphism f factors through $h^{(t+1)}$.

(*ii*) $f_t = \alpha_r g_{t-r}$, where $|g_{t-r}| = r - t \, (0 \leq r \leq t-1)$. In these cases, the differentials

$$d(f_t) = d(\alpha_r)g_{t-r} + (-1)^r\alpha_r d(g_{t-r}) = [\alpha_0][y_{red}^{r-1}]g_{t-r} + \ldots + [\alpha_{r-1}][y_{red}^0]g_{t-r} + (-1)^r\alpha_r d(g_{t-r}),$$

which implies that $[f_{t-1}] = -[y_{red}^{r-1}]g_{t-r}, \ldots, [f_{t-r}] = -[y_{red}^0]g_{t-r}$ and $[f_{t-r-1}] = (-1)^{r+1}d(g_{t-r})$. Then we have that

$$\begin{pmatrix} f_t \\ [f_{t-1}] \\ \cdots \\ [f_1] \\ [f_0] \end{pmatrix} = \begin{pmatrix} \alpha_r g_{t-r} \\ -[y_{red}^{r-1}]g_{t-r} \\ \cdots \\ -[y_{red}^0]g_{t-r} \\ (-1)^{r+1}d(g_{t-r}) \\ 0 \\ \cdots \\ 0 \end{pmatrix} = d_{RA_t} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ (-1)^t g_{t-r} \\ 0 \\ \cdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ (-1)^t g_{t-r} \\ 0 \\ \cdots \\ 0 \end{pmatrix} d_L$$

is a zero element in $\mathrm{Hom}_{\mathcal{D}}(L, RA_t)$. Therefore, the morphism $h^{(t+1)}$ is a minimal right $(\mathrm{add}A)$-approximation of $RA_t (1 \le t \le m)$. Hence, the isomorphism $\Sigma^{-t-1}\varepsilon_{\le t+1}Y \simeq RA_{t+1}$ holds. $\quad\square$

We further assume that the zeroth homology $H^0\Pi$ is finite dimensional. Then the dg algebra Π satisfies Assumptions 2.1 and moreover it is strongly $(m+2)$-Calabi-Yau.

Since the simple module S_i is zero in the generalized m-cluster category $\mathcal{C}_\Pi = \mathrm{per}\Pi/\mathcal{D}_{fd}(\Pi)$, the corresponding minimal cofibrant resolution Y also becomes zero in $\mathcal{C}_\Pi$. Taking truncations of Y, we obtain $m+2$ triangles in $\mathcal{C}_\Pi$

$$\pi(\varepsilon_{\le t}Y) \longrightarrow 0 \longrightarrow \pi(\varepsilon_{\ge t+1}Y) \longrightarrow \Sigma\pi(\varepsilon_{\le t}Y), \qquad 0 \le t \le m+1,$$

where $\pi : \mathrm{per}\Pi \to \mathcal{C}_\Pi$ is the canonical projection functor. Therefore, the following theorem holds:

Theorem 4.11. *Under the assumptions in Proposition 4.10 and the assumption that $H^0\Pi$ is finite dimensional, the image of RA_t is isomorphic to the image of LA_{m+1-t} in the generalized m-cluster category $\mathcal{C}_\Pi$ for each integer $0 \le t \le m+1$.*

Proof. The following isomorphisms

$$\pi(RA_t) \simeq \pi(\Sigma^{-t}\varepsilon_{\le t}Y) \simeq \pi(\Sigma^{-t-1}\varepsilon_{\ge t+1}Y) \simeq \pi(LA_{m+1-t})$$

are true in $\mathcal{C}_\Pi$ for all integers $0 \le t \le m+1$. $\quad\square$

In the presence of loops, the objects RA_t and LA_r do not always satisfy the relations in Theorem 4.11. See the following example.

Example 4.12. Suppose that m is 2. Let Q be the quiver whose vertex set Q_0 has only one vertex '•' and whose arrow set Q_1 has two loops α and β of degree -1. Then the completed deformed preprojective dg algebra $\Pi = \widehat{\Pi}(Q, 4, 0)$ has the underlying graded quiver as follows

with $|\alpha| = |\beta| = -1$ and $|t| = -3$. The differential takes the following values

$$d(\alpha) = 0 = d(\beta), \quad d(t) = 2\alpha^2 + 2\beta^2.$$

The algebra Π is an indecomposable object in the derived category $\mathcal{D}(\Pi)$. Let $P = \Pi$. Then we have the equality $\Pi = P \oplus M$, where $M = 0$. Then LA_r is isomorphic to $\Sigma^r P$ and RA_r is isomorphic to $\Sigma^{-r}P$ for all $r \ge 0$.

The zeroth homology $H^0\Pi$ is one-dimensional and generated by the trivial path $e_\bullet$. Let $\mathcal{C}_\Pi$ be the generalized 2-cluster category. We claim that the image of RA_1 in $\mathcal{C}_\Pi$ is not isomorphic to the image of LA_2. Otherwise, assume that $\pi(RA_1)$ is isomorphic to $\pi(LA_2)$. Then the following isomorphisms hold

$$\mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_2), \Sigma\pi(LA_2)) \simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_2), \Sigma\pi(RA_1)) \simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\Sigma^2 P, \Sigma\pi(\Sigma^{-1}P))$$

$$\simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\Sigma^2 P, P) \simeq \mathrm{Hom}_{\mathcal{D}(\Pi)}(\Sigma^2 P, P) \simeq H^{-2}\Pi.$$

The left end term of these isomorphisms vanishes since $\pi(LA_2)$ is a 2-cluster tilting object, while the right end term is a 3-dimensional space whose basis is $\{\alpha^2, \alpha\beta, \beta\alpha\}$. Therefore, we obtain a contradiction.

5. Periodicity property

Lemma 5.1. *Let A be a dg algebra satisfying Assumptions 2.1. Let x and y be two integers satisfying $x \le y + m + 1$. Suppose that the object X lies in $\mathcal{D}^{\le x} \cap \mathrm{per}A$ and the object Y lies in $^\perp\mathcal{D}^{\le y} \cap \mathrm{per}A$. Then the quotient functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ induces an isomorphism*

$$\mathrm{Hom}_\mathcal{D}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi(X), \pi(Y)).$$

Proof. This proof is quite similar to the proof of Lemma 2.9 given in [28].

First, we show the injectivity.

Assume that $f : X \to Y$ is a morphism in $\mathcal{D}$ whose image in $\mathcal{C}_A$ is zero. It follows that f factors through some N in $\mathcal{D}_{fd}(A)$. Let $f = hg$. Consider the following diagram
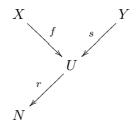

We have that g factors through $\tau_{\le x}N$ because $X \in \mathcal{D}^{\le x}$ and $\mathrm{Hom}_\mathcal{D}(\mathcal{D}^{\le x}, \tau_{\ge x+1}N)$ vanishes.

Now since $\tau_{\le x}N$ is still in $\mathcal{D}_{fd}(A)$, by the Calabi-Yau property, the following isomorphism

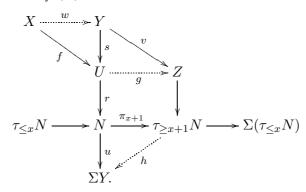$$D\mathrm{Hom}_\mathcal{D}(\tau_{\le x}N, Y) \simeq \mathrm{Hom}_\mathcal{D}(Y, \Sigma^{m+2}(\tau_{\le x}N))$$

holds. Since $\Sigma^{m+2}(\tau_{\le x}N)$ belongs to $\mathcal{D}^{\le x-m-2}(\subseteq \mathcal{D}^{\le y-1})$, the right hand side of the above isomorphism is zero. Therefore, the morphism f is zero in the derived category $\mathcal{D}$.

Second, we show the surjectivity.
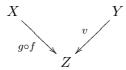
Consider an arbitrary fraction $s^{-1}f$ in $\mathcal{C}_A$

where the cone N of s is in $\mathcal{D}_{fd}(A)$. Now look at the following diagram

By the Calabi-Yau property, the space $\mathrm{Hom}_\mathcal{D}(\tau_{\le x}N, \Sigma Y)$ is isomorphic to $D\mathrm{Hom}_\mathcal{D}(Y, \Sigma^{m+1}(\tau_{\le x}N))$, which is zero since $x - m - 1 \le y$. Thus, there exists a morphism h such that $u = h \circ \pi_{x+1}$. Now we embed h into a triangle in $\mathcal{D}$ as follows

$$Y \xrightarrow{v} Z \longrightarrow \tau_{\ge x+1}N \xrightarrow{h} \Sigma Y.$$

It follows that the morphism v factors through s by some morphism g. Then we can get a new fraction

$$
\begin{array}{ccc}
X & & Y \\
& {}_{g \circ f}\searrow \quad {}^{v}\swarrow & \\
& Z &
\end{array}
$$

where the cone of v is $\tau_{\geq x+1}N (\in \mathcal{D}_{fd}(A))$. This fraction is equal to the one we start with because

$$v^{-1}(g \circ f) = (g \circ s)^{-1}(g \circ f) \sim s^{-1}f.$$

Moreover, since the space $\operatorname{Hom}_{\mathcal{D}}(X, \tau_{\geq x+1}N)$ vanishes, there exists a morphism $w : X \to Y$ such that $g \circ f = v \circ w$. Therefore, the fraction above is exactly the image of w in $\operatorname{Hom}_{\mathcal{D}}(X, Y)$ under the quotient functor π. $\square$

Note that in the assumptions of the above lemma, we do not necessarily suppose that the objects X and Y lie in some shifts of the fundamental domain.

A special case of Lemma 5.1 is that, if X lies in $\mathcal{D}^{\leq m} \cap \operatorname{per}A$, then the quotient functor $\pi : \operatorname{per}A \to \mathcal{C}_A$ induces an isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(X, RA_t) \simeq \operatorname{Hom}_{\mathcal{C}_A}(\pi(X), \pi(RA_t))$$

for any nonnegative integer t, where RA_t belongs to ${}^{\perp}\mathcal{D}^{\leq -1}$.

Theorem 5.2. *Under the assumptions of Theorem 4.11, for each positive integer t,*
  1) *the image of RA_t is isomorphic to the image of $RA_{t(\operatorname{mod} m+1)}$ in $\mathcal{C}_{\Pi}$,*
  2) *the image of LA_t is isomorphic to the image of $LA_{t(\operatorname{mod} m+1)}$ in $\mathcal{C}_{\Pi}$.*

Proof. We only show the first statement. Then the second one can be obtained similarly.

Following Theorem 4.11, the image of RA_{m+1} in $\mathcal{C}_{\Pi}$ is isomorphic to P, which is RA_0 by definition. Let us denote '$t\,(\operatorname{mod} m+1)$' by $\bar{t}$. We prove the statement by induction.

Assume that the image of RA_t is isomorphic to the image of $RA_{\bar{t}}$ in $\mathcal{C}_{\Pi}$. Consider the following two triangles in $\mathcal{D}(\Pi)$

$$RA_{t+1} \longrightarrow A^{(t+1)} \overset{f^{(t+1)}}{\longrightarrow} RA_t \longrightarrow \Sigma RA_{t+1},$$

$$RA_{\overline{t+1}} \longrightarrow A^{(\overline{t+1})} \overset{f^{(\overline{t+1})}}{\longrightarrow} RA_{\bar{t}} \longrightarrow \Sigma RA_{\overline{t+1}},$$

and also consider their images in $\mathcal{C}_{\Pi}$. By Lemma 5.1, the isomorphism

$$\operatorname{Hom}_{\mathcal{D}(\Pi)}(L, RA_t) \simeq \operatorname{Hom}_{\mathcal{C}_{\Pi}}(L, \pi(RA_t))$$

holds for any object $L \in \operatorname{add}M$ and any nonnegative integer t. Hence, the images $\pi(f^{(t+1)})$ and $\pi(f^{(\overline{t+1})})$ are minimal right $(\operatorname{add}M)$-approximations of $\pi(RA_t)$ and $\pi(RA_{\bar{t}})$ in $\mathcal{C}_{\Pi}$, respectively. By hypothesis, $\pi(RA_t)$ is isomorphic to $\pi(RA_{\bar{t}})$. Therefore, the objects $A^{(t+1)}$ and $A^{(\overline{t+1})}$ are isomorphic, and $\pi(RA_{t+1})$ is isomorphic to $\pi(RA_{\overline{t+1}})$ in $\mathcal{C}_{\Pi}$. This completes the statement. $\square$

Remark 5.3. Section 10 in [18] gave a class of $(2n+1)$-Calabi-Yau (only for even integers $2n$, not for all integers $m \geq 2$) triangulated categories (arising from certain Cohen-Macaulay rings) which contain infinitely many indecomposable $2n$-cluster tilting objects.

In the following, for every integer $m \geq 2$, we construct an $(m+1)$-Calabi-Yau triangulated category which contains infinitely many indecomposable m-cluster tilting objects.

When $m = 2$, we use the same quiver Q as in Example 4.12.

When $m > 2$, let Q be the quiver consisting of one vertex $\bullet$ and one loop α of degree -1.

Let $\Pi = \widehat{\Pi}(Q, m+2, 0)$ be the associated completed deformed preprojective dg algebra. Clearly, Π is an indecomposable object in the derived category $\mathcal{D}(\Pi)$, the zeroth homology $H^0\Pi$ is one-dimensional and the path α^s is a nonzero element in the homology $H^{-s}\Pi$ ($s \in \mathbb{N}^*$). Let $\mathcal{C}_{\Pi}$ be the generalized m-cluster category and $\pi : \operatorname{per}\Pi \to \mathcal{C}_{\Pi}$ the canonical projection functor. Set $P = \Pi$. Then $\Pi = P \oplus 0$. For each integer $t \geq 0$, the object LA_t is isomorphic to $\Sigma^t P$ and the object RA_t is isomorphic to $\Sigma^{-t}P$. Now we claim that

1) For any two integers $r > t \geq 0$, the object $\pi(RA_r)$ is not isomorphic to $\pi(RA_t)$ in $\mathcal{C}_\Pi$, and the object $\pi(LA_r)$ is not isomorphic to $\pi(LA_t)$ in $\mathcal{C}_\Pi$.
2) For any two integers $r_1, r_2 \geq 0$, the objects $\pi(RA_{r_1})$ and $\pi(LA_{r_2})$ are not isomorphic in $\mathcal{C}_\Pi$.

Otherwise, similarly as in Example 4.12, the following contradictions will appear

$$(0 =) \operatorname{Hom}_{\mathcal{C}_\Pi}(\pi(RA_t), \Sigma\pi(RA_t)) = \operatorname{Hom}_{\mathcal{C}_\Pi}(\pi(RA_t), \Sigma\pi(RA_r)) \simeq \operatorname{Hom}_{\mathcal{C}_\Pi}(\Sigma^{-t}P, \Sigma^{1-r}P)$$

$$\simeq \operatorname{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{t-r+1}P) \simeq \operatorname{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{t-r+1}P) \simeq H^{t-r+1}\Pi \, (\neq 0);$$

$$(0 =) \operatorname{Hom}_{\mathcal{C}_\Pi}(\pi(LA_r), \Sigma\pi(LA_r)) = \operatorname{Hom}_{\mathcal{C}_\Pi}(\pi(LA_r), \Sigma\pi(LA_t)) \simeq \operatorname{Hom}_{\mathcal{C}_\Pi}(\Sigma^{r}P, \Sigma^{t+1}P)$$

$$\simeq \operatorname{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{t-r+1}P) \simeq \operatorname{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{t-r+1}P) \simeq H^{t-r+1}\Pi \, (\neq 0);$$

$$(0 =) \operatorname{Hom}_{\mathcal{C}_\Pi}(\pi(LA_{r_2}), \Sigma\pi(LA_{r_2})) = \operatorname{Hom}_{\mathcal{C}_\Pi}(\pi(LA_{r_2}), \Sigma\pi(RA_{r_1})) \simeq \operatorname{Hom}_{\mathcal{C}_\Pi}(\Sigma^{r_2}P, \Sigma^{1-r_1}P)$$

$$\simeq \operatorname{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{1-r_1-r_2}P) \simeq \operatorname{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{1-r_1-r_2}P) \simeq H^{1-r_1-r_2}\Pi \, (\neq 0);$$

where the left end terms become zero, the right end terms are nonzero since $t - r + 1 \leq 0$ and $1 - r_1 - r_2 < 0$, and the isomorphism

$$\operatorname{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{-s}P) \simeq \operatorname{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{-s}P)$$

holds for any $s \in \mathbb{N}$ following Lemma 5.1.

Therefore, the $(m+1)$-Calabi-Yau triangulated category $\mathcal{C}_\Pi$ contains infinitely many m-cluster tilting objects, and the objects $\pi(RA_t)$ and $\pi(LA_r)$ do not satisfy the relations in Theorem 4.11 and Theorem 5.2 in the presence of loops.

6. AR $(m+3)$-ANGLES RELATED TO P-INDECOMPOSABLES

Let $\mathcal{T}$ be an additive Krull-Schmidt category. We denote by $J_\mathcal{T}$ the *Jacobson radical* [3] of $\mathcal{T}$. Let $f \in \mathcal{T}(X, Y)$ be a morphism. Then f is called (in [18]) a *sink map* of $Y \in \mathcal{T}$ if f is right minimal, $f \in J_\mathcal{T}$, and

$$\mathcal{T}(-, X) \xrightarrow{f \cdot} J_\mathcal{T}(-, Y) \longrightarrow 0$$

is exact as functors on $\mathcal{T}$. The definition of *source maps* is given dually.

Let n be a positive integer. Given n triangles in a triangulated category,

$$X_{i+1} \xrightarrow{b_{i+1}} B_i \xrightarrow{a_i} X_i \to \Sigma X_{i+1}, \quad 0 \leq i < n,$$

the complex

$$X_n \xrightarrow{b_n} B_{n-1} \xrightarrow{b_{n-1}a_{n-1}} B_{n-2} \to \ldots \to B_1 \xrightarrow{b_1 a_1} B_0 \xrightarrow{a_0} X_0$$

is called (in [18]) an $(n+2)$-*angle*.

Definition 6.1 ([18]). Let H be an m-cluster tilting object in a Krull-Schmidt triangulated category. We call an $(m+3)$-angle with X_0, X_{m+1} and all $B_i (0 \leq i \leq m)$ in $\text{add}H$ an *AR* $(m+3)$-*angle* if the following conditions are satisfied

a) a_0 is a sink map of X_0 in $\text{add}H$ and b_{m+1} is a source map of X_{m+1} in $\text{add}H$, and
b) a_i (resp. b_i) is a minimal right (resp. left) $(\text{add}H)$-approximation of X_i for each integer $1 \leq i \leq m$.

Remark 6.2. An AR $(m+3)$-angle with right term X_0 (resp. left term X_{m+1}) depends only on X_0 (resp. X_{m+1}) and is unique up to isomorphism as a complex.

We will use the AR angle theory to show the following theorem, which gives a more virtual criterion than Theorem 5.8 in [18] for our case.

Theorem 6.3. *Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ and i a vertex of Q. Assume that the zeroth homology $H^0\Pi$ is finite dimensional and there are no loops of Q at vertex i. Then the almost complete m-cluster tilting P-object $\Pi/e_i\Pi$ has exactly $m+1$ complements in the generalized m-cluster category $\mathcal{C}_\Pi$.*

Proof. Set $RA_0 = P_i = e_i\Pi$ and $M = \Pi/e_i\Pi$. Section 4 gives us a construction of iterated mutations RA_t of P_i in the derived category $\mathcal{D}(\Pi)$, that is, the morphism $h^{(1)} : P_0' \to P_i$ is a minimal right $(\mathrm{add}M)$-approximation of P_i, and morphisms $h^{(t+1)} : P_t' \to RA_t$ $(1 \leq t \leq m)$ are minimal right $(\mathrm{add}A)$-approximations of RA_t with P_t' in $\mathrm{add}M$. Let $\mathcal{A}$ (resp. $\mathcal{M}$) denote the subcategory $\mathrm{add}\pi(\Pi)$ (resp. $\mathrm{add}\pi(M)$) in the generalized m-cluster category $\mathcal{C}_\Pi$.

Step 1. Since P_0', P_i and M are in the fundamental domain, the morphism $h^{(1)}$ can be viewed as a minimal right $\mathcal{M}$-approximation in $\mathcal{C}_\Pi$, that is, the sequence

$$\mathcal{A}(-, P_0')|_{\mathcal{M}} \xrightarrow{h^{(1)} \cdot} \mathcal{A}(-, P_i)|_{\mathcal{M}} = J_{\mathcal{A}}(-, P_i)|_{\mathcal{M}} \to 0,$$

is exact as functors on $\mathcal{M}$. Since there are no loops of $\overline{Q}^V$ of degree zero at vertex i, the Jacobson radical of $\mathrm{End}_{\mathcal{A}}(P_i)$ $(\simeq \mathrm{End}_{\mathcal{D}(\Pi)}(P_i))$ consists of combinations of cyclic paths $p = a_1 \ldots a_r$ $(r \geq 2)$ of $\overline{Q}^V$ of degree zero. The path p factors though $e_{s(a_1)}\Pi$ and factors through $h^{(1)}$. Therefore, we have an exact sequence

$$\mathcal{A}(P_i, P_0') \xrightarrow{h^{(1)} \cdot} \mathrm{rad}\,\mathrm{End}_{\mathcal{A}}(P_i) \longrightarrow 0.$$

Thus, the morphism $h^{(1)}$ is a sink map in the subcategory $\mathcal{A}$.

Step 2. The morphisms $h^{(t+1)}$ $(1 \leq t \leq m)$ are minimal right $(\mathrm{add}A)$-approximations of RA_t with P_t' in $\mathrm{add}M$. Since the objects $RA_t (1 \leq t \leq m)$ and P_t' lie in the shift $\Sigma^{-m}\mathcal{F}$ of the fundamental domain by Proposition 2.8, the images of $h^{(t+1)}$ are minimal right $\mathcal{A}$-approximations in $\mathcal{C}_\Pi$.

Step 3. Consider the morphisms $\alpha^{(t)}$ in the triangles of constructing RA_t in $\mathcal{D}(\Pi)$

$$\Sigma^{-1}RA_{t-1} \longrightarrow RA_t \xrightarrow{\alpha^{(t)}} P_{t-1}' \xrightarrow{h^{(t)}} RA_{t-1}, \quad 1 \leq t \leq m.$$

We already know that the maps $\alpha^{(t)}$ are minimal left $(\mathrm{add}M)$-approximations in $\mathcal{D}(\Pi)$. Now applying the functor $\mathrm{Hom}_{\mathcal{D}(\Pi)}(-, P_i)$ to the above triangles, we obtain long exact sequences

$$\ldots \to \mathrm{Hom}_{\mathcal{D}(\Pi)}(P_{t-1}', P_i) \longrightarrow \mathrm{Hom}_{\mathcal{D}(\Pi)}(RA_t, P_i) \longrightarrow \mathrm{Hom}_{\mathcal{D}(\Pi)}(\Sigma^{-1}RA_{t-1}, P_i) \to \ldots.$$

The terms $\mathrm{Hom}_{\mathcal{D}(\Pi)}(\Sigma^{-1}RA_{t-1}, P_i)$ are zero since all RA_{t-1} lie in $^{\perp}\mathcal{D}(\Pi)^{\leq -1}$. Hence, the morphisms $\alpha^{(t)}$ are minimal left $(\mathrm{add}A)$-approximations in $\mathcal{D}(\Pi)$. Since the objects $RA_t(1 \leq t \leq m)$ and P_t' lie in the shift $\Sigma^{-m}\mathcal{F}$, the images of $\alpha^{(t)}$ are minimal left $\mathcal{A}$-approximations in $\mathcal{C}_\Pi$.

Step 4. Consider the following two triangles in $\mathcal{D}(\Pi)$

$$RA_{m+1} \xrightarrow{\alpha^{(m+1)}} P_m' \xrightarrow{h^{(m+1)}} RA_m \longrightarrow \Sigma RA_{m+1},$$

$$P_i \xrightarrow{g^{(1)}} P_m' \xrightarrow{\beta^{(1)}} LA_1 \longrightarrow \Sigma P_i.$$

Since the objects P_i, P_m' and LA_1 are in the fundamental domain $\mathcal{F}$, the second triangle can also be viewed as a triangle in $\mathcal{C}_\Pi$ and the morphism $\beta^{(1)}$ is a minimal right $\mathcal{M}$-approximation of LA_1. Note that the objects RA_m and P_m' belong to $\Sigma^{-m}\mathcal{F}$. Hence, the image of the first triangle

$$\pi(RA_{m+1}) \xrightarrow{\pi(\alpha^{(m+1)})} P_m' \xrightarrow{\pi(h^{(m+1)})} \pi(RA_m) \longrightarrow \Sigma\pi(RA_{m+1})$$

is a triangle in $\mathcal{C}_\Pi$ with $\pi(h^{(m+1)})$ a minimal right $\mathcal{M}$-approximation of $\pi(RA_m)$. By Theorem 4.11, the image of RA_m is isomorphic to the image of LA_1 in $\mathcal{C}_\Pi$. Thus, the images of these two triangles in $\mathcal{C}_\Pi$ are isomorphic. We can also check that $g^{(1)}$ is a source map in $\mathcal{A}$ as Step 1. Therefore, the image $\pi(\alpha^{(m+1)})$ is also a source map in $\mathcal{A}$ with $\pi(RA_{m+1})$ isomorphic to P_i in $\mathcal{C}_\Pi$.

Step 5. Now we form the following $(m+3)$-angle in $\mathcal{C}_\Pi$

$$P_i = \pi(RA_{m+1}) \xrightarrow{\varphi_{m+1}} P_m' \xrightarrow{\varphi_m} P_{m-1}' \longrightarrow \ldots \longrightarrow P_1' \xrightarrow{\varphi_1} P_0' \xrightarrow{\varphi_0} P_i,$$

where φ_0 is equal to $\pi(h^{(1)})$, the morphism φ_t $(1 \leq t \leq m)$ is the composition $\pi(\alpha^{(t)})\pi(h^{(t+1)})$, and φ_{m+1} is equal to $\pi(\alpha^{(m+1)})$. From the above four steps, we know that φ_0 is a sink map in $\mathcal{A}$, and φ_{m+1} is a source map in $\mathcal{A}$. Furthermore, this $(m+3)$-angle is the AR $(m+3)$-angle determined by P_i. Since the indecomposable object P_i does not belong to $\mathrm{add}(\oplus_{t=0}^m P_t')$, following Theorem 5.8 in [18], the almost complete m-cluster tilting P-object $\Pi/e_i\Pi$ has exactly $m+1$ complements $e_i\Pi$, $\pi(RA_1), \ldots, \pi(RA_m)$ in $\mathcal{C}_\Pi$. The proof is completed. $\qquad\square$

7. Liftable almost complete m-cluster tilting objects for strongly $(m+2)$-Calabi-Yau case

Keep the assumptions as in Theorem 6.3. Let $\Pi = \widehat{\Pi}(Q, m+2, W)$. Let Y be a liftable almost complete m-cluster tilting object in the generalized m-cluster category $\mathcal{C}_\Pi$. Assume that Z is a basic cofibrant silting object in $\mathrm{per}\Pi$ such that $\pi(Z/Z')$ is isomorphic to Y, where $\pi : \mathrm{per}\Pi \to \mathcal{C}_\Pi$ is the canonical projection and Z' is an indecomposable direct summand of Z. Let A be the dg endomorphism algebra $\mathrm{Hom}_\Pi^\bullet(Z, Z)$ and F the left derived functor $- \overset{L}{\otimes}_A Z$. From the proof of Theorem 3.3, we know that F is a Morita equivalence from $\mathcal{D}(A)$ to $\mathcal{D}(\Pi)$ and A satisfies Assumptions 2.1. We denote the truncated dg subalgebra $\tau_{\leq 0}A$ by E. Since A has its homology concentrated in nonpositive degrees, the canonical inclusion $E \hookrightarrow A$ is a quasi-isomorphism. Then the left derived functor $- \overset{L}{\otimes}_E A$ is a Morita equivalence from $\mathcal{D}(E)$ to $\mathcal{D}(A)$.

Theorem 7.1 ([20]). *Let l be a commutative ring. Let B and B' be two dg l-algebras and X a dg B-B'-bimodule which is cofibrant over B. Assume that B and B' are flat as dg l-modules and*

$$- \overset{L}{\otimes}_{B'} X : \mathcal{D}(B') \to \mathcal{D}(B)$$

is an equivalence. Then the dg algebras B and B' have isomorphic cyclic homology and isomorphic Hochschild homology.

A corollary of Theorem 7.1 is that B' is strongly $(m+2)$-Calabi-Yau if and only if so is B.

The object Z is canonically an k-module, so the dg algebras A and E are k-algebras. Thus, the derived equivalent dg algebras Π, A and E are flat as dg k-modules. Following Remark 4.4 and Theorem 7.1, the dg algebras A and E are also strongly $(m+2)$-Calabi-Yau.

We will show that the dg algebra E satisfies the assumption in Theorem 4.7, that is E lies in $PCAlgc(l')$ for some finite dimensional separable commutative k-algebra l'. In fact, $l' = \prod_{|Z|} k$, where $|Z|$ is the number of indecomposable direct summands of Z in $\mathrm{per}\Pi$. Furthermore, from the following lemma, we can deduce that $l' = l$.

Lemma 7.2. *Suppose that B is a dg algebra with positive homologies being zero. Then all basic cofibrant silting objects have the same number of indecomposable direct summands in $\mathrm{per}B$.*

Proof. The triangulated category $\mathrm{per}B$ contains an additive subcategory $\mathcal{B} := \mathrm{add}B$. Since the dg algebra B has its homology concentrated in nonpositive degrees, it follows that

$$\mathrm{Hom}_{\mathrm{per}B}(\mathcal{B}, \Sigma^p\mathcal{B}) = 0, \quad p > 0.$$

Since the category $\mathrm{per}B$, which consists of the compact objects in $\mathcal{D}(B)$, and the category $\mathrm{add}B$ are both idempotent split, by Proposition 5.3.3 of [7], the isomorphism

$$K_0(\mathrm{per}B) \simeq K_0(\mathrm{add}B)$$

holds, where $K_0(-)$ denotes the Grothendieck group.

Let Z be any basic cofibrant silting object in $\mathrm{per}B$ and B' its dg endomorphism algebra $\mathrm{Hom}_B^\bullet(Z, Z)$. Then B' has its homology concentrated in nonpositive degrees and $\mathrm{per}B'$ is triangle equivalent to $\mathrm{per}B$. Therefore, we have $K_0(\mathrm{per}B') \simeq K_0(\mathrm{add}B')$ and $K_0(\mathrm{per}B') \simeq K_0(\mathrm{per}B)$. As a consequence, the following isomorphisms hold

$$K_0(\mathrm{add}B) \simeq K_0(\mathrm{add}B') \simeq K_0(\mathrm{add}Z).$$

Thus, any basic cofibrant silting object in per$B$ has the same number of indecomposable direct summands as that of the dg algebra B itself. $\qquad\square$

When forgetting the grading, the dg algebra E becomes to be $E^u := Z^0A \oplus (\prod_{r<0} A^r)$, where $Z^0A (= \text{Hom}_{\mathcal{C}(\Pi)}(Z, Z))$ consists of the zeroth cycles of A. For any $x \in \prod_{r<0} A^r$, the element $1 + x$ clearly has an inverse element. It follows that $\prod_{r<0} A^r$ is contained in $\text{rad}(E^u)$. We have the following canonical short exact sequence

$$0 \to B^0A \to Z^0A \xrightarrow{p} H^0A \to 0,$$

where B^0A is a two-sided ideal of the algebra Z^0A consisting of the zeroth boundaries of A.

Following from Lemma 4.9, without loss of generality, we can assume that the basic silting object Z is a minimal perfect dg Π-module.

Lemma 7.3. *Keep the above notation and suppose that Z is a minimal perfect dg Π-module. Then B^0A lies in the radical of Z^0A.*

Proof. Let f be an element in B^0A. Then f is of the form $d_Z h + h d_Z$ for some degree -1 morphism $h : Z \to Z$. Since Z is minimal perfect, the entries of f lie in the ideal $\mathfrak{m}$ generated by the arrows of $\widetilde{Q}^V$. Then for any morphism $g : Z \to Z$, the morphism $1_Z - gf$ admits an inverse $1_Z + gf + (gf)^2 + \ldots$. Similarly for the morphism $1_Z - fg$. It follows that f lies in the radical of the algebra Z^0A. This completes the proof. $\qquad\square$

The epimorphism p in the above short exact sequence induces an epimorphism

$$\overline{p} : Z^0A/\text{rad}(Z^0A) \to H^0A/\text{rad}(H^0A).$$

Since B^0A lies in the radical of Z^0A, the epimorphism $\overline{p}$ is an isomorphism. Therefore, the following isomorphisms

$$E^u/\text{rad}(E^u) \simeq Z^0A/\text{rad}(Z^0A) \simeq H^0A/\text{rad}(H^0A)$$

are true. Note that perΠ is Krull-Schmidt and Hom-finite. Since the algebra $E_i := \text{End}_{\text{per}\Pi}(Z_i)$ is local and k is algebraically closed, the quotient $E_i/\text{rad}(E_i)$ is isomorphic to k. Then we have that

$$H^0A/\text{rad}(H^0A) \simeq \text{End}_{\text{per}\Pi}(Z)/\text{rad}(\text{End}_{\text{per}\Pi}(Z)) \simeq \prod_{|Z|} E_i/\text{rad}(E_i) \simeq \prod_{|Z|} k \, (= l).$$

Hence, the dg algebra E lies in $PCAlgc(l)$. Therefore, E is quasi-isomorphic to some good completed deformed preprojective dg algebra $\widehat{\Pi}(Q', m+2, W')$ (denoted by Π'). Moreover, $H^0\Pi'$ is equal to H^0A which is finite dimensional.

The following diagram

$$
\begin{array}{ccccccc}
\text{per}\Pi' & \xrightarrow{-\overset{L}{\otimes}_{\Pi'} E} & \text{per}E & \xrightarrow{-\overset{L}{\otimes}_E A} & \text{per}A & \xrightarrow{-\overset{L}{\otimes}_A Z} & \text{per}\Pi \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}_{\Pi'} & \longrightarrow & \mathcal{C}_E & \longrightarrow & \mathcal{C}_A & \longrightarrow & \mathcal{C}_\Pi
\end{array}
$$

is commutative, where each functor in the rows is an equivalence and each functor in a column is the canonical projection. The preimage of Z in perΠ', under the equivalence F given by the composition of the functors in the top row, is Π'. Let $\Pi'_0 = e_j \Pi'$ be the P-indecomposable dg Π'-module such that $F(\Pi'_0) = Z'$ in perΠ, where j is a vertex of Q'. Assume that there are no loops of Q' at vertex j. It follows from Theorem 6.3 that the almost complete m-cluster tilting P-object Π'/Π'_0 has exactly $m + 1$ complements in $\mathcal{C}_{\Pi'}$. Note that the image of Π'/Π'_0 in $\mathcal{C}_\Pi$, under the equivalence given by the composition of the functors in the bottom row, is Y. Therefore, the liftable almost complete m-cluster tilting object Y has exactly $m+1$ complements in $\mathcal{C}_\Pi$.

As a conclusion, we write down the following theorem.

Theorem 7.4. *Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ whose zeroth homology $H^0\Pi$ is finite dimensional. Let Z be a basic silting object in $\mathrm{per}\Pi$ which is minimal perfect and cofibrant. Denote by E the dg algebra $\tau_{\leq 0}(\mathrm{Hom}^\bullet_\Pi(Z, Z))$. Then*

1) *E is quasi-isomorphic to some good completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', m+2, W')$, where the quiver Q' has the same number of vertices as Q and $H^0\Pi'$ is finite dimensional;*

2) *let Y be a liftable almost complete m-cluster tilting object of the form $\pi(Z/Z')$ in $\mathcal{C}_\Pi$ for some indecomposable direct summand Z' of Z. If we further assume that there are no loops at the vertex j of Q', where $e_j\Pi'\overset{L}{\otimes}_{\Pi'}Z = Z'$, then Y has exactly $m+1$ complements in $\mathcal{C}_\Pi$.*

Here we would like to point out a special case of the above theorem, namely $m = 1$ and $Z = LA_1^{(k)}$ with respect to some vertex k of Q. Let $(Q^\star, W^\star)$ denote the (reduced) mutation $\mu_k(Q, W)$ defined in [11] of the quiver with potential (Q, W) at vertex k. Let A be the dg endomorphism algebra $\mathrm{Hom}^\bullet_\Pi(Z, Z)$ and $\Pi^\star$ the good completed deformed preprojective dg algebra $\widehat{\Pi}(Q^\star, m+2, W^\star)$. By [26], there is a canonical morphism from Π^* to A. Define three functors as follows:

$$F = -\overset{L}{\otimes}_{\Pi^\star} Z, \quad F_1 = -\overset{L}{\otimes}_{\Pi^\star} A, \quad F_2 = -\overset{L}{\otimes}_A Z.$$

Clearly, we have that $F = F_2 F_1$ and F_2 is a quasi-inverse equivalence. It was shown in [26] that F is a quasi-inverse equivalence. The following isomorphisms

$$H^n(\Pi^\star) \simeq \mathrm{Hom}_{\Pi^\star}(\Pi^\star, \Sigma^n\Pi^\star) \simeq \mathrm{Hom}_A(A, \Sigma^n A) \simeq H^n A$$

become true, which implies that $\Pi^\star$ and A are quasi-isomorphic. Therefore, the quiver with potential (Q', W') appearing in Theorem 7.4 1) for this special case can be chosen as $\mu_k(Q, W)$.

As the end part of this section, we state a 'reasonable' conjecture about the non-loop assumption in the above theorem for completed deformed preprojective dg algebras.

Definition 7.5. Let r be a positive integer. An algebra $A \in PCAlgc(l)$ is said to be *r-rigid* if

$$HH_0(A) \simeq l, \quad \text{and} \quad HH_p(A) = 0 \ (1 \leq p \leq r-1),$$

where $HH_*(A)$ is the pseudo-compact version of the Hochschild homology of the dg algebra A.

Remark 7.6. For completed Ginzburg algebras associated to quivers with potentials, our definition of 1-rigidity coincides with the definition of rigidity in [11]. Proposition 8.1 in [11] states that any rigid reduced quiver with potential is 2-acyclic. Then no loops will be produced following their mutation rule. Although we do not know whether the quiver Q' related to such a silting object as in Theorem 7.4 can be obtained from mutation of quivers with potentials, we can still obtain that the quiver Q' always does not contain loops in the condition of 1-rigidity (see Corollary 7.9).

Proposition 7.7. *The completed deformed preprojective dg algebras $\Pi = \widehat{\Pi}(Q, m+2, 0)$ associated to acyclic quivers Q are m-rigid.*

Proof. It is clear that the zeroth component Π^0 of Π is just the finite dimensional path algebra kQ (denoted by B) and the $(-p)$th component of Π is zero for $1 \leq p \leq m-1$. Thus, the Hochschild homology of Π is given by

$$HH_0(\Pi) = B/[B, B] = \prod_{|Q_0|} k,$$

$$HH_p(\Pi) = HH_p(B) = ker(\partial_p^0)/\mathrm{Im}(\partial_{p+1}^0) \quad (1 \leq p \leq m-1),$$

where $\partial_p^0 : B^{\otimes_k(p+1)} \to B^{\otimes_k p}$ is the pth row differential of the uppermost row in the Hochschild complex $X := \Pi \overset{L}{\otimes}_{\Pi^e} \Pi$.

Since the path algebra kQ is of finite dimension and of finite global dimension and k is algebraically closed, we have $HH_p(B) = 0$ for all integers $p > 0$, cf. Proposition 2.5 of [20]. It follows that the dg algebra $\widehat{\Pi}(Q, m + 2, 0)$ is m-rigid. $\qquad\square$

Proposition 7.8. *Let $\Pi = \widehat{\Pi}(Q, m + 2, W)$ be a good completed deformed preprojective dg algebra and p a fixed integer in the segment $[0, m]$. Suppose the p-th Hochschild homology of Π satisfies the isomorphism*

$$HH_p(\Pi) \simeq \begin{cases} \prod_{|Q_0|} k & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

Then $\overline{Q}^V$ does not contain loops with zero differential and of degree $-p$.

Proof. Let a be a loop of $\overline{Q}^V$ at some vertex i with zero differential and of degree $-p$. The element a lies in the rightmost column of the Hochschild complex X of Π. By assumption the differential $d(a)$ is zero, so a is an element in $HH_p(\Pi)$. Now we claim that a is a nonzero element in $HH_p(\Pi)$.

First, the superpotential W is a linear combination of paths of length at least 3, so $d(\widetilde{Q}_1^V) \subseteq \mathfrak{m}^2$, where $\mathfrak{m}$ is the two-sided ideal of Π generated by the arrows of $\widetilde{Q}^V$. Second, it is obvious that the relation $\text{Im}\partial_1 \cap \{\text{loops of } \widetilde{Q}^V\} = \emptyset$ holds. Therefore, the loop a can not be written in the form $\sum d(\gamma) + \sum \partial_1(u \otimes v)$ for paths $\gamma \in e_i\mathfrak{m}e_i$ and u, v paths of $\widetilde{Q}^V$, which means that a is a nonzero element in $HH_p(\Pi)$.

Note that the trivial paths associated to the vertices of Q are nonzero elements in $HH_0(\Pi)$. Hence, we get a contradiction to the isomorphism in the assumption. As a result, the quiver $\overline{Q}^V$ does not contain loops with zero differential and of degree $-p$. $\qquad\square$

Corollary 7.9. *Keep the notation as in Theorem 7.4 and let $m = 1$. Suppose that Π is 1-rigid. Then the new quiver Q' does not contain loops.*

Proof. It follows from statement 1) in Theorem 7.4 that E is quasi-isomorphic to some good completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', 3, W')$. Then following Theorem 7.1 and the analysis before Theorem 7.4, we can obtain that the dg algebras Π and Π' have isomorphic Hochschild homology. Therefore, the new dg algebra Π' is also 1-rigid. Note that every arrow of Q' has zero degree and thus has zero differential. Hence, by Proposition 7.8 the quiver Q' does not contain loops. $\qquad\square$

Conjecture 7.10. *Let $\Pi = \widehat{\Pi}(Q, m + 2, W)$ be an m-rigid good completed deformed preprojective dg algebra whose zeroth homology $H^0\Pi$ is finite dimensional. Then any liftable almost complete m-cluster tilting object has exactly $m + 1$ complements in $\mathcal{C}_\Pi$.*

Following the same procedure as in the proof of Corollary 7.9, we know that the good completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', m + 2, W')$ in Theorem 7.4 is also m-rigid, and the new quiver Q' does not contain loops of degree zero. It seems that we would like to get a stronger result than Proposition 7.8, that is, m-rigidity implies that $\overline{Q'}^V$ does not contain loops (not only loops with zero differential). If this is true, then it follows from statement 2) in Theorem 7.4 that any liftable almost complete m-cluster tilting object has exactly $m + 1$ complements in $\mathcal{C}_\Pi$.

If Conjecture 7.10 holds, then the m-rigidity property shown in Proposition 7.7 of the dg algebra $\Pi = \widehat{\Pi}(Q, m + 2, 0)$ with Q an acyclic quiver implies that any liftable almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ has exactly $m + 1$ complements. Later Proposition 8.6 shows that any almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ is liftable in the 'acyclic quiver' case. Thus, on one hand, if Conjecture 7.10 holds, we can deduce a common result both in [31] and [32], namely, any almost complete m-cluster tilting object in the classical m-cluster category $\mathcal{C}_Q^{(m)}$ has exactly $m + 1$ complements. On the other hand, it follows from this common result for the classical m-cluster category $\mathcal{C}_Q^{(m)}$, which is triangle equivalent to the corresponding generalized

m-cluster category $\mathcal{C}_\Pi$, that any almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ should have exactly $m + 1$ complements.

8. A long exact sequence and the acyclic case

Let A be a dg algebra satisfying Assumptions 2.1. In the first part of this section, we give a long exact sequence to see the relations between extension spaces in generalized m-cluster categories $\mathcal{C}_A$ and extension spaces in derived categories $\mathcal{D}(= \mathcal{D}(A))$. If the extension spaces between two objects of $\mathcal{C}_A$ are zero, in some cases, we can deduce that the extension spaces between these two objects are also zero in the derived category $\mathcal{D}$.

Proposition 8.1. *Suppose that X and Y are two objects in the fundamental domain $\mathcal{F}$. Then there is a long exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathcal{D}}(X, Y) \to \operatorname{Ext}^1_{\mathcal{C}_A}(X, Y) \to D\operatorname{Ext}^m_{\mathcal{D}}(Y, X)$$
$$\to \operatorname{Ext}^2_{\mathcal{D}}(X, Y) \to \operatorname{Ext}^2_{\mathcal{C}_A}(X, Y) \to D\operatorname{Ext}^{m-1}_{\mathcal{D}}(Y, X)$$
$$\to \quad \cdots \qquad \cdots \quad \to$$
$$\operatorname{Ext}^m_{\mathcal{D}}(X, Y) \to \operatorname{Ext}^m_{\mathcal{C}_A}(X, Y) \to D\operatorname{Ext}^1_{\mathcal{D}}(Y, X) \to 0.$$

Proof. We have the canonical triangle

$$\tau_{\leq -m} X \to X \to \tau_{\geq 1-m} X \to \Sigma(\tau_{\leq -m} X),$$

which yields the long exact sequence

$$\cdots \to \operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m} X), Y) \to \operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t} X, Y) \to \operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -m} X), Y) \to \cdots, \quad t \in \mathbb{Z}.$$

Step 1. *The isomorphism*

$$\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m} X), Y) \simeq D\operatorname{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t} X)$$

holds when $t \leq m + 1$.

By the Calabi-Yau property, there holds the isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m} X), Y) \simeq D\operatorname{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t}(\tau_{\geq 1-m} X)), \quad t \in \mathbb{Z}. \qquad (8.1).$$

Applying the functor $\operatorname{Hom}_{\mathcal{D}}(Y, -)$ to the triangle which we start with, we obtain the exact sequence

$$(Y, \Sigma^{m+2-t}(\tau_{\leq -m} X)) \to (Y, \Sigma^{m+2-t} X) \to (Y, \Sigma^{m+2-t}(\tau_{\geq 1-m} X)) \to (Y, \Sigma^{m+3-t}(\tau_{\leq -m} X)),$$

where $(-, -)$ denotes $\operatorname{Hom}_{\mathcal{D}}(-, -)$. When $t \leq m+1$, we have that $(-m) - (m+2-t) \leq -m-1$. Then the objects $\Sigma^{m+2-t}(\tau_{\leq -m} X)$ and $\Sigma^{m+3-t}(\tau_{\leq -m} X)$ belong to $\mathcal{D}^{\leq -m-1}$. Note that Y is in $^\perp \mathcal{D}^{\leq -m-1}$. Therefore, the following isomorphism holds

$$\operatorname{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t}(\tau_{\geq 1-m} X)) \simeq \operatorname{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t} X). \qquad (8.2).$$

As a consequence, when $t \leq m + 1$, together by (8.1) and (8.2), we have the isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m} X), Y) \simeq D\operatorname{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t} X).$$

Moreover, if $t \leq 1$, the object $\Sigma^{m+2-t} X$ belongs to $\mathcal{D}^{\leq -m-1}$, so the space $\operatorname{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t} X)$ vanishes, and so does the space $\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m} X), Y)$.

Step 2. *When $t \leq m$, we have the following isomorphism*

$$\operatorname{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -m} X), Y) \simeq \operatorname{Hom}_{\mathcal{C}_A}(\pi X, \Sigma^t(\pi Y)).$$

Consider the triangles

$$\tau_{\leq s-1} X \to \tau_{\leq s} X \to \Sigma^{-s}(H^s X) \to \Sigma(\tau_{\leq s-1} X), \qquad s \in \mathbb{Z}.$$

Applying the functor $\operatorname{Hom}_{\mathcal{D}}(-, Y)$ to these triangles, we can obtain the following long exact sequences

$$\cdots \to (\Sigma^{-s-t}(H^s X), Y) \to (\Sigma^{-t}(\tau_{\leq s} X), Y) \to (\Sigma^{-t}(\tau_{\leq s-1} X), Y) \to (\Sigma^{-s-t-1}(H^s X), Y) \to \cdots,$$

where $(-,-)$ denotes $\mathrm{Hom}_{\mathcal{D}}(-,-)$. Using the Calabi-Yau property, we have that

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-s-t}(H^s X), Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-s-t}(H^s X)), \qquad t \in \mathbb{Z}.$$

When $t \leq -s$, the inequality $m+2-s-t-1 \geq m+1$ holds. So the two objects $\Sigma^{m+2-s-t}(H^s X)$ and $\Sigma^{m+2-s-t-1}(H^s X)$ belong to $\mathcal{D}^{\leq -m-1}$. Therefore, $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-s-t}(H^s X), Y)$ and the space $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-s-t-1}(H^s X), Y)$ are zero, and the following isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq s} X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq s-1} X), Y)$$

holds. As a consequence, we can get the following isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -t} X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -t-1} X), Y) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -m} X), Y),\ t \leq m. \ (8.3).$$

Since the functor $\pi : \mathrm{per} A \to \mathcal{C}_A$ induces an equivalence from $\Sigma^t \mathcal{F}$ to $\mathcal{C}$ (Proposition 3.1 applies to shifted t-structure), the following bijections are true

$$\mathrm{Hom}_{\mathcal{C}_A}(\pi X, \Sigma^t(\pi Y)) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi(\tau_{\leq -t} X), \pi(\Sigma^t Y)) \simeq \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -t} X, \Sigma^t Y). \qquad (8.4).$$

Hence, when $t \leq m$, together by (8.3) and (8.4), we have the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -m} X), Y) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi X, \Sigma^t(\pi Y)).$$

Therefore, the long exact sequence at the beginning becomes

$$0 = \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-1}(\tau_{\geq 1-m} X), Y) \to \mathrm{Ext}^1_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^m_{\mathcal{D}}(Y,X)$$

$$\to \mathrm{Ext}^2_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^2_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^{m-1}_{\mathcal{D}}(Y,X)$$

$$\to \quad \cdots \quad \cdots \quad \to$$

$$\mathrm{Ext}^m_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^m_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^1_{\mathcal{D}}(Y,X) \to \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-m-1} X, Y) = 0.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remarks 8.2. 1) When $m = 1$, the long exact sequence in Proposition 8.1 becomes the following short exact sequence (already appearing in the proof of Proposition 3.5)

$$0 \to \mathrm{Ext}^1_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^1_{\mathcal{D}}(Y,X) \to 0 \qquad (8.5),$$

which was presented in [2] for the Hom-finite 2-Calabi-Yau case, and also was presented in [28] for the Jacobi-infinite 2-Calabi-Yau case.

2) If T is an object in the fundamental domain $\mathcal{F}$ satisfying

$$\mathrm{Ext}^i_{\mathcal{D}}(T,T) = 0, \quad i = 1, \ldots, m,$$

then the long exact sequence in Proposition 8.1 implies that the spaces $\mathrm{Ext}^i_{\mathcal{C}_A}(T,T)$ also vanish for integers $1 \leq i \leq m$.

Suppose that X and Y are two objects in the fundamental domain. It is clear that $\mathrm{Ext}^i_{\mathcal{D}}(X,Y)$ vanishes when $i > m$, since X belongs to $\mathcal{F}$ and $\Sigma^i Y$ lies in $\mathcal{D}^{\leq -m-1}$. Now we assume that the spaces $\mathrm{Ext}^i_{\mathcal{C}_A}(X,Y)$ are zero for integers $1 \leq i \leq m$. What about the extension spaces $\mathrm{Ext}^i_{\mathcal{D}}(X,Y)$ in the derived category? Do they always vanish?

When $m = 1$, the short exact sequence (8.5) implies that the space $\mathrm{Ext}^1_{\mathcal{D}}(X,Y)$ vanishes.

When $m > 1$, we will give the answer for completed Ginzburg dg categories (the same as completed deformed preprojective dg algebras in this case) arising from acyclic quivers.

Proposition 8.3. *Let Q be an acyclic quiver. Let Γ be the completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q,0)$ and $\mathcal{C}_\Gamma$ the generalized m-cluster category. Suppose that X and Y are two objects in the fundamental domain $\mathcal{F}$ which satisfy*

$$\mathrm{Ext}^i_{\mathcal{C}_\Gamma}(X,Y) = 0, \quad i = 1, \ldots, m.$$

Then the extension spaces $\mathrm{Ext}^i_{\mathcal{D}(\Gamma)}(X,Y)$ vanish for all positive integers i.

Proof. Let B be the path algebra kQ and Ω the inverse dualizing complex $R\mathrm{Hom}_{B^e}(B, B^e)$. Set $\Theta = \Sigma^{m+1}\Omega$. Then the $(m+2)$-Calabi-Yau completion [24] of B is the tensor dg category

$$\Pi_{m+2}(B) = T_B(\Theta) = B \oplus \Theta \oplus (\Theta \otimes_B \Theta) \oplus \dots.$$

Theorem 6.3 in [24] shows that $\Pi_{m+2}(B)$ is quasi-isomorphic to the completed Ginzburg dg category Γ. Thus, we can write Γ as

$$\Gamma = B \oplus \Theta \oplus (\Theta \overset{L}{\otimes}_B \Theta) \oplus \dots = \oplus_{p \geq 0}\Theta^{\overset{L}{\otimes}_B p}.$$

Let X', Y' be two objects in $\mathcal{D}_{fd}(B)$. The following isomorphisms hold

$$\mathrm{Hom}_{\mathcal{D}(\Gamma)}(X' \overset{L}{\otimes}_B \Gamma, Y' \overset{L}{\otimes}_B \Gamma) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(X', Y' \overset{L}{\otimes}_B \Gamma|_B) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(X', Y' \overset{L}{\otimes}_B (\oplus_{p \geq 0}\Theta^{\overset{L}{\otimes}_B p}))$$

$$\simeq \mathrm{Hom}_{\mathcal{D}(B)}(X', \oplus_{p \geq 0}(Y' \overset{L}{\otimes}_B \Theta^{\overset{L}{\otimes}_B p})) \simeq \oplus_{p \geq 0}\mathrm{Hom}_{\mathcal{D}(B)}(X', Y' \overset{L}{\otimes}_B \Theta^{\overset{L}{\otimes}_B p}).$$

By Lemma 2.3, the category $\mathcal{D}_{fd}(B)$ admits a Serre functor S whose inverse is $- \overset{L}{\otimes}_B \Omega$. Therefore, the functor $- \overset{L}{\otimes}_B \Theta$ is equal to the functor $S^{-1}\Sigma^{m+1} (\simeq \tau^{-1}\Sigma^m)$, where τ is the Auslander-Reiten translation. As a consequence, we have that

$$\mathrm{Hom}_{\mathcal{D}(\Gamma)}(X' \overset{L}{\otimes}_B \Gamma, Y' \overset{L}{\otimes}_B \Gamma) \simeq \oplus_{p \geq 0}\mathrm{Hom}_{\mathcal{D}_{fd}(B)}(X', (\tau^{-1}\Sigma^m)^p Y').$$

Let $\mathcal{C}_Q^{(m)}$ be the m-cluster category $\mathcal{D}_{fd}(B)/(\tau^{-1}\Sigma^m)^{\mathbb{Z}}$. Consider the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}_{fd}(B) & \xrightarrow{\;-\overset{L}{\otimes}_B\Gamma\;} & \mathrm{per}\Gamma \\
{\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi_\Gamma} \\
\mathcal{C}_Q^{(m)} & \xrightarrow[\simeq]{\;-\overset{L}{\otimes}_B\Gamma\;} & \mathcal{C}_\Gamma.
\end{array}
$$

Under the equivalence, let $X = X' \overset{L}{\otimes}_B \Gamma$ and $Y = Y' \overset{L}{\otimes}_B \Gamma$, so the vanishing of spaces $\mathrm{Ext}^i_{\mathcal{C}_\Gamma}(X, Y)$ implies that $\mathrm{Ext}^i_{\mathcal{C}_Q^{(m)}}(X', Y')$ also vanish for integers $1 \leq i \leq m$. Note that

$$\mathrm{Ext}^i_{\mathcal{C}_Q^{(m)}}(X', Y') \simeq \oplus_{p \in \mathbb{Z}}\mathrm{Ext}^i_{\mathcal{D}_{fd}(B)}(X', (\tau^{-1}\Sigma^m)^p Y').$$

Hence, we obtain that

$$\mathrm{Ext}^i_{\mathcal{D}(\Gamma)}(X, Y) \simeq \oplus_{p \geq 0}\mathrm{Ext}^i_{\mathcal{D}_{fd}(B)}(X', (\tau^{-1}\Sigma^m)^p Y') = 0, \qquad 1 \leq i \leq m.$$

$\square$

Let Q be an ordinary acyclic quiver and B the path algebra kQ. Let Γ be its completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q, 0)$. Let T be an m-cluster tilting object in $\mathcal{C}_Q^{(m)}$. Then T is induced from an object T' (that is, $T = \pi(T')$) in the fundamental domain

$$\mathcal{S}_m := \mathcal{S}_m^0 \vee \Sigma^m B, \quad \text{where } \mathcal{S}_m^0 := \mathrm{mod}B \vee \Sigma(\mathrm{mod}B) \dots \vee \Sigma^{m-1}(\mathrm{mod}B).$$

Lemma 8.4 ([9])**.** *The object T' is a partial silting object, that is,*

$$\mathrm{Hom}_{\mathcal{D}_{fd}(B)}(T', \Sigma^i T') = 0, \qquad i > 0;$$

and T' is maximal with this property.

An object in $\mathcal{D}_{fd}(B)$ which satisfies the 'maximal partial silting' property as in Lemma 8.4 is called a 'silting' object in [9]. Next we will show that our definition for silting object in $\mathrm{per}B$ coincides with their definition.

Lemma 8.5. *Let U be a basic partial silting object in $\mathcal{D}_{fd}(B)$. Then U is maximal partial silting if and only if U generates $\mathrm{per}B$.*

Proof. On one hand, assume that U is a basic partial silting object and generates per$B$. By Lemma 7.2, the object U has the same number of indecomposable direct summands as that of the dg algebra B itself. That is, U is a basic partial silting object with $|Q_0|$ indecomposable direct summands. Following from Lemma 2.2 in [9], we obtain that U is a maximal partial silting object.

On the other hand, assume that U is a maximal partial silting object in $\mathcal{D}_{fd}(B)$. We decompose U into a direct sum $\Sigma^{k_1} U_1 \oplus \ldots \oplus \Sigma^{k_r} U_r$ such that each U_i lies in mod$B$ and $k_1 < \ldots < k_r$. Set $U' = \oplus_{i=1}^r U_i$. It follows from Lemma 2.2 in [9] that the object U' can be ordered to a complete exceptional sequence. Let $C(U')$ be the smallest full subcategory of mod$B$ which contains U' and is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms. By Lemma 3 in [6], the subcategory $C(U')$ is equal to mod$B$. As a consequence, the object U generates $\mathcal{D}_{fd}(B)$ which is equal to per$B$. $\square$

Since B is finite dimensional and hereditary, the subcategory $\mathcal{S}_m^0$ is contained in $^{\perp}\mathcal{D}(B)^{\leq -m-1}$. The isomorphism
$$\mathrm{Hom}_{\mathcal{D}(B)}(\Sigma^m B, M) \simeq H^m M \quad (M \in \mathcal{D}(B))$$
implies that $\Sigma^m B$ is in $^{\perp}\mathcal{D}(B)^{\leq -m-1}$. So $\mathcal{S}_m$ is contained in $\mathcal{D}(B)^{\leq 0} \cap^{\perp} \mathcal{D}(B)^{\leq -m-1} \cap \mathcal{D}_{fd}(B)$.

Set $Z = T' \overset{L}{\otimes}_B \Gamma$. For any object N in $\mathcal{D}(\Gamma)$, we have the following canonical isomorphism
$$\mathrm{Hom}_{\mathcal{D}(\Gamma)}(T' \overset{L}{\otimes}_B \Gamma, N) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(T', \mathrm{RHom}_\Gamma(\Gamma, N)).$$
When N lies in $\mathcal{D}(\Gamma)^{\leq -m-1}$, the right hand side of the above isomorphism becomes zero. Thus, the object Z is in the fundamental domain of $\mathcal{D}(\Gamma)$. The spaces $\mathrm{Ext}^i_{\mathcal{C}_Q^{(m)}}(T, T)$ vanish for integers $1 \leq i \leq m$, following the proof of Proposition 8.3, the space $\mathrm{Hom}_{\mathcal{D}(\Gamma)}(Z, \Sigma^i Z)$ is zero for each positive integer i. In addition, Lemma 8.4 and Lemma 8.5 together imply that T' generates $\mathcal{D}_{fd}(B)$. Hence, the object Z generates perΓ. So Z is a basic silting object whose image in $\mathcal{C}_\Gamma$ is $T \overset{L}{\otimes}_B \Gamma$.

Now we conclude the above analysis to get the following proposition.

Proposition 8.6. *Let Q be an acyclic quiver and B its path algebra. Let Γ be the completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q, 0)$ and $\mathcal{C}_\Gamma$ the generalized m-cluster category. Then any m-cluster tilting object in $\mathcal{C}_\Gamma$ is induced by a silting object in $\mathcal{F}$ under the canonical projection $\pi : \mathrm{per}\Gamma \to \mathcal{C}_\Gamma$.*

Proof. Let $\overline{T}$ be an m-cluster tilting object in $\mathcal{C}_\Gamma$. Then $\overline{T}$ can be written as $T \overset{L}{\otimes}_B \Gamma$ for some m-cluster tilting object T in $\mathcal{C}_Q^{(m)}$, where T is induced by some silting object T' in $\mathcal{D}_{fd}(B)$. The object $T' \overset{L}{\otimes}_B \Gamma$ (denoted by Z) is a silting object in the fundamental domain $\mathcal{F}(\subseteq \mathrm{per}\Gamma)$ whose image under the canonical projection $\pi : \mathrm{per}\Gamma \to \mathcal{C}_\Gamma$ is equal to $\overline{T}$. This completes the proof. $\square$

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